Please work out the ten (10) problems stated below – Show work and explain reasoning.

All rvs are defined on the same probability triple \((\Omega, \mathcal{F}, \mathbb{P})\).

1. Consider the discrete rv \(X : \Omega \rightarrow \mathbb{R}\) with support \(S_X = \mathbb{Z}\) and pmf \(p_X \equiv (p_X(x), x \in \mathbb{Z})\) given by
   \[
p_X(x) = \frac{C}{1 + |x|^2}, \quad x = 0, \pm 1, \pm 2, \ldots
   \]
   for some \(C > 0\).
   a. How should \(C > 0\) be determined?
   b. Does \(\mathbb{E}[X]\) exist?

2. Consider a rv \(X : \Omega \rightarrow \mathbb{R}\) which is \textbf{uniform} over the interval \((-a, a)\) where \(a > 0\).
   a. Give its probability distribution function \(F_X : \mathbb{R} \rightarrow [0, 1]\).
   b. Find the probability distribution function \(F_{X^+} : \mathbb{R} \rightarrow [0, 1]\) of the rv \(X^+ = \max(0, X)\). Does the probability distribution function of the rv \(X^+\) admit a density? Is the rv \(X^+\) a discrete rv?

3. Let \(X_1, \ldots, X_n\) be \(n\) discrete rvs \(\Omega \rightarrow \mathbb{N}\) defined on the same probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). They are assumed to be \textbf{mutually independent}. Define the sum rv \(S_n \equiv X_1 + \ldots + X_n\).
   a. Compute the pmf of the rv \(S_n\) if for all \(k = 1, \ldots, n\), \(X_k \sim \text{Ber}(p)\) for some \(0 < p < 1\).
   b. Compute the pmf of the rv \(S_n\) if for all \(k = 1, \ldots, n\), \(X_k \sim \text{Bin}(n_k, p)\) for some \(0 < p < 1\) and positive integer \(n_k\). Can you use Part a to conclude without having to do any calculations?
   c. Compute the pmf of the rv \(S_n\) if for all \(k = 1, \ldots, n\), \(X_k \sim \text{Poi}(\lambda_k)\) for some \(\lambda_k > 0\).

4. 
This problem deals with the following random experiment: A coin is tossed infinitely many times under identical and independent conditions. It is assumed that on a single toss the likelihood of head is \( p \) (with \( 0 < p < 1 \)). To model this experiment use the probability model developed in Section 6.2 (where \( \Omega = \{0,1\}^{\mathbb{N}} \)).

**a.** The mapping \( X : \Omega \to \mathbb{N} \cup \{+\infty\} \) is defined by

\[
X(\omega) \equiv \begin{cases} 
\text{The number of tosses before the first Head appears in the sample } \omega, & \omega \in \Omega \\
+\infty & \text{if Head never appears in the sample } \omega.
\end{cases}
\]

with \( X(\omega) = +\infty \) if Head never appears in the sample \( \omega \). Explain why the mapping \( X : \Omega \to \mathbb{R} \) so defined is indeed a rv. Is it a discrete rv?

**b.** Find the pmf of this rv, i.e., \( \{P[X = m], m = 1, 2, \ldots\} \).

**c.** On the probability triple used here, is it possible to define a rv \( Y : \Omega \to \mathbb{R} \) which is not a discrete rv? In the affirmative give an example.

5. Let \( \{W_k, k = 1, 2, \ldots\} \) denote a collection of mutually independent Walsh rvs with same parameter \( p \) (in \((0,1)) \) all defined on the same probability triple \((\Omega, \mathcal{F}, \mathbb{P})\), i.e., for each \( k = 1, \ldots, \) we have

\[
\mathbb{P}[W_k = w] = \begin{cases} 
p & \text{if } w = 1 \\
1 - p & \text{if } w = -1.
\end{cases}
\]

For each \( k = 1, \ldots, \), write \( W_k^\star \) for the product of the \( k \) rvs \( W_1, \ldots, W_k \), i.e., \( W_k^\star \equiv \prod_{\ell=1}^{n} W_\ell \).

**a.** For each \( k = 1, 2, \ldots, \), compute \( \mathbb{E}[W_k^\star] \).

**b.** Is the rv \( W_k^\star \) independent of the \( \{-1,1\}^k \)-valued rv \( (W_1, \ldots, W_k) \)?

**b.** Is the rv \( W_k^\star \) independent of the \( \{-1,1\}^{k-1} \)-valued rv \( (W_2, \ldots, W_k) \)?

6. Let \( N \) be a Poisson rv, and let \( \{B_n, n = 1, 2, \ldots\} \) be a collection of Bernoulli rvs with

\[
\mathbb{P}[B_n = 1] = 1 - \mathbb{P}[B_n = 0] = p, \quad n = 1, 2, \ldots \quad 0 < p < 1.
\]

If the rvs \( \{N, B_n, n = 1, 2 \ldots\} \) are mutually independent, show that the rvs \( X \) and \( Y \) defined by

\[
X \equiv \sum_{i=1}^{N} B_i \quad \text{and} \quad Y \equiv \sum_{i=1}^{N} (1 - B_i)
\]

are independent rvs with \( X \) and \( Y \) Poisson rvs with parameters \( \lambda p \) and \( \lambda (1 - p) \), respectively. Can you use this result to provide an alternative solution to Exercise 3.c? Explain! Again a case of probabilistic reasoning at work!

7. Consider the discrete rv \( Z : \Omega \to \mathbb{Z} \) whose pmf \( p_Z = (p_Z(z), z \in \mathbb{Z}) \) (under \( \mathbb{P} \)) is given by

\[
p_Z(z) = C q^{|z|}, \quad z \in \mathbb{Z}
\]
Define the discrete rvs \( X \) whenever the series \( \sum_{\ell=0}^{\infty} x^{\ell} \) converges, namely \( |x| < 1 \).

8. Let \( B \) and \( X \) be two independent rvs \( \Omega \to \mathbb{N} \) with \( B \sim \text{Ber}(\frac{1}{2}) \) and \( X \sim \text{Geo}(p) \) with \( 0 < p < 1 \), i.e., \( \mathbb{P}[B = 1] = p \) and \( \mathbb{P}[B = 0] = 1 - p \), while \( \mathbb{P}[X = \ell] = p(1-p)^{\ell-1} \) for \( \ell = 1, 2, \ldots \). Define the rv \( Y : \Omega \to [0, +\infty) \) given by

\[
Y \equiv B \cdot X + (1-B) \cdot \frac{1}{X}
\]

with the understanding that \( \frac{1}{0} = +\infty \).

a. Determine its support \( S_Y \) of the discrete rv \( Y \) and find its pmf \( p_Y \).

b. Introduce the discrete rv \( Z : \Omega \to [0, +\infty) \) given by \( Z \equiv Y^{-1} \). Determine the support \( S_Z \) of the discrete rv \( Z \) and find its pmf \( p_Z \).

c. Compute \( \mathbb{E}[Y] \) [HINT: Recall from Real Analysis that \( \int_0^a (\sum_{\ell=0}^{\infty} x^{\ell}) \, dx = \ldots \) whenever the series \( \sum_{\ell=0}^{\infty} x^{\ell} \) converges, namely \( |x| < 1 \)].

9. Let \( X_1, \ldots, X_n \) be \( n \) discrete rvs \( \Omega \to \mathbb{N} \) defined on the same probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). They are assumed to be mutually independent and to be geometrically distributed in the sense that for each \( k = 1, \ldots, n \), we have \( X_k \sim \text{Geo}(p_k) \) for some \( 0 < p_k < 1 \) (not necessarily identical).

a. For each \( k = 1, 2, \ldots, n \), compute \( \mathbb{P}[X_k > x] \) for each \( x = 0, 1, \ldots \).

b. Find the pmf of the rv \( V_n \equiv \min\{X_1, \ldots, X_n\} \) [HINT: Compute \( \mathbb{P}[V_n > x] \) for each \( x = 0, 1, \ldots \) and identify the pmf!].

b. Compute \( \mathbb{E}[V_n] \).

10. Let \( P, U_1, \ldots, U_n \) be \( n+1 \) mutually independent rvs defined on the same probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). Assume that the rvs \( U_1, \ldots, U_n \) are uniformly distributed on the interval \([0, 1]\), and that the rv \( P \) is simple rv of the form

\[
P = \sum_{i \in I} p_i 1[A_k]
\]

for some finite \( \mathcal{F} \)-partition \( \{A_i, \ i \in I\} \) and scalars \( \{p_i, \ i \in I\} \) in \([0, 1]\) all distinct. Define the discrete rvs \( X_1, \ldots, X_n \) to be

\[
X_k \equiv 1[U_k \leq P], \quad k = 1, 2, \ldots
\]

a. Assume first that \(|I| = 1\). What is the common pmf of the rvs \( X_1, \ldots, X_n \)? Are the rvs \( X_1, \ldots, X_n \) pairwise independent? Are they mutually independent?

Assume next that \(|I| \geq 2\) with \( \{p_i, \ i \in I\} \subset (0, 1) \).
b. Find the common pmf of the rvs $X_1, \ldots, X_n$.

c. Are the rvs $X_1, \ldots, X_n$ pairwise independent? Are they mutually independent?

d. Evaluate the expectations $\mathbb{E}[X_k]$ and $\mathbb{E}[X_kX_\ell]$ for all $k, \ell = 1, 2, \ldots, n$ in terms of expected values associated with the rv $P$. 