ENEE 621: ESTIMATION AND DETECTION THEORY

Midterm Examination I: Solutions

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Problem 1

[1a] Since

$$\frac{\partial}{\partial\theta}\ln f_\theta(y) = \frac{2(y-\theta)}{2\sigma^2},$$

we have

$$M(\theta) = \mathrm{E}_\theta\left[\frac{(Y-\theta)^2}{\sigma^4}\right] = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}.$$

[1b] The ML estimate of θ on the basis of $y^n = (y_1, \cdots, y_n)$ is $g_{ML}(y^n) = \frac{\sum_{i=1}^n y_i}{n}$. The error covariance of g_{ML} is

$$\begin{aligned}\Sigma_\theta\left(g_{ML}\right) &= \mathrm{E}_\theta\left[\left(g_{ML}(Y^n)-\theta\right)^2\right] \\ &= \mathrm{E}_\theta\left[\left(\frac{1}{n}\sum_{i=1}^n Y_i - \theta\right)^2\right] \\ &= \frac{1}{n^2}\mathrm{E}_\theta\left[\left(\sum_{i=1}^n Y_i - n\theta\right)^2\right] \\ &= \frac{1}{n^2}n\sigma^2 = \frac{\sigma^2}{n}.\end{aligned}$$

Noting that $M^{(n)}(\theta) = \frac{n}{\sigma^2}$, it follows that g_{ML} is efficient since $\Sigma_\theta(g_{ML}) = M^{(n)}(\theta)^{-1}$.

Alternatively, the efficiency of g_{ML} also follows upon verifying that

$$g_{ML}(y^n) - \theta = M^{(n)}(\theta)^{-1}\frac{\partial}{\partial\theta}f_\theta(y^n).$$

Problem 2

[2a] Since

$$f_\theta(y_1, \cdots, y_n) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!} = \frac{e^{-n\theta}\theta^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i!},$$

we have for $\tilde{\theta} \triangleq e^{-\theta}$ that

$$f_{\tilde{\theta}}(y^n) = \frac{\tilde{\theta}^n \left(\ln \frac{1}{\tilde{\theta}}\right)^{\sum_{i=1}^n y_i}}{\prod_{i=1}^n y_i!},$$

where $\tilde{\theta} \in (0, 1)$.

Next,

$$\ln f_{\tilde{\theta}}(y^n) = n \ln \tilde{\theta} + \sum_{i=1}^{n} y_i \ln \left(\ln \frac{1}{\tilde{\theta}}\right) - \sum_{i=1}^{n} \ln y_i!,$$

so that

$$\frac{\partial}{\partial \tilde{\theta}} \ln f_{\tilde{\theta}}(y^n) = \frac{n}{\tilde{\theta}} + (\sum_{i=1}^{n} y_i) \frac{1}{(\ln \frac{1}{\tilde{\theta}})} (-\frac{1}{\tilde{\theta}}),$$

which, upon being equated to 0, gives $\tilde{\theta} = e^{-\frac{\sum_{i=1}^n y_i}{n}}$.

Hence, $\tilde{\theta}_{ML}(y^n) = e^{-\frac{\sum_{i=1}^n y_i}{n}}$, and exists provided $\sum_{i=1}^n y_i \neq 0$.

[2b] Since $e^{-\theta} = P_\theta(Y_1 = 0)$, we see that the estimator g defined by

$$g(y^n) = 1(y_1 = 0)$$

is an unbiased estimator of $e^{-\theta}$ as

$$\mathrm{E}_\theta\left[g(Y^n)\right] = P_\theta(Y_1 = 0) = e^{-\theta}.$$

Next, we know that the statistic T_n defined by

$$T_n(y^n) \triangleq \sum_{i=1}^{n} y_i$$

is a sufficient statistic for θ and, hence, also for $e^{-\theta}$. Also, since $\sum_{i=1}^n Y_i$ is a Poisson rv with mean $n\theta$, T_n is a complete sufficient statistic for θ and, hence, also for $e^{-\theta}$. Consequently, the estimator $\tilde{g}oT_n$ is a MVUE for $e^{-\theta}$, where

$$\begin{aligned}\tilde{g}(t) =& \mathrm{E}_\theta\left[g(Y^n)|T_n = t\right] \\ =& P_\theta(Y_1 = 0|\sum_{i=1}^{n} Y_i = t) \\ =& \frac{P_\theta(Y_1 = 0)P_\theta(\sum_{i=2}^{n} Y_i = t)}{P_\theta(\sum_{i=1}^{n} Y_i = t)} \\ =& \frac{e^{-\theta}\frac{e^{-(n-1)\theta}((n-1)\theta)^t}{t!}}{e^{-n\theta}\frac{(n\theta)^t}{t!}} \\ =& \left(1 - \frac{1}{n}\right)^t.\end{aligned}$$

Hence, the desired MVUE is

$$\left(\tilde{g}oT_n\right)(y^n) = \left(1 - \frac{1}{n}\right)^{\sum_{i=1}^{n} y_i}.$$

Problem 3

[3a] We have

$$\begin{aligned} M(\theta) &= \mathbf{E}_\theta \left[|\frac{\partial}{\partial\theta} \ln f_\theta(Y)|^2 \right] \\ &= \mathbf{E}_\theta \left[\left(\frac{1}{\theta} - Y \right)^2 \right] = \frac{1}{\theta^2}. \end{aligned}$$

[3b] Let A denote the observed event. Then,

$$\begin{aligned} P_\theta(A) &= \binom{M}{m} \left(P_\theta(Y > y_o)\right)^m \left(P_\theta(Y \le y_o)\right)^{M-m} \\ &= \binom{M}{m} \left(e^{-\theta y_o}\right)^m \left(1 - e^{-\theta y_o}\right)^{M-m} \end{aligned}$$

whence,

$$\ln P_\theta(A) = \ln \binom{M}{m} - \theta y_o m + (M - m) \ln \left(1 - e^{-\theta y_o}\right).$$

Upon setting $\frac{\partial}{\partial\theta} \ln P_\theta(A) = 0$, we get

$$\theta = \frac{1}{y_o} \ln \frac{M}{m}.$$

Thus, $\theta_{ML}(M, m, y_o) = \frac{1}{y_o} \ln \frac{M}{m}$, which exists provided $m < M$.