ENEE 621: ESTIMATION AND DETECTION THEORY

Midterm Examination II: Solutions

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Problem 1

It is easily seen that

$$P(\theta = -1|Y = y) = \frac{f_{-1}(y) \cdot \frac{1}{2}}{f(y)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}} \cdot \frac{1}{2}}{\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{(y+1)^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-1)^2}{2}}\right]}$$

$$= \frac{e^{-\frac{(y+1)^2}{2}}}{e^{-\frac{(y+1)^2}{2}} + e^{-\frac{(y-1)^2}{2}}}$$

and

$$P(\theta = 1|Y = y) = \frac{e^{-\frac{(y-1)^2}{2}}}{e^{-\frac{(y+1)^2}{2}} + e^{-\frac{(y-1)^2}{2}}}$$

Hence,

$$P(\theta = -1|Y = y) < P(\theta = 1|Y = y), \quad y > 0$$

$$\text{and } P(\theta = -1|Y = y) \geq P(\theta = 1|Y = y), \quad y \leq 0$$

whence $G_\theta(t|Y = y)$ has the form as in Fig. 1:

Thus,

$$g_{MEM}(y) = \begin{cases} -1, & y < 0 \\ 1, & y > 0 \\ \text{any value in } [-1, 1], & y = 0. \end{cases}$$

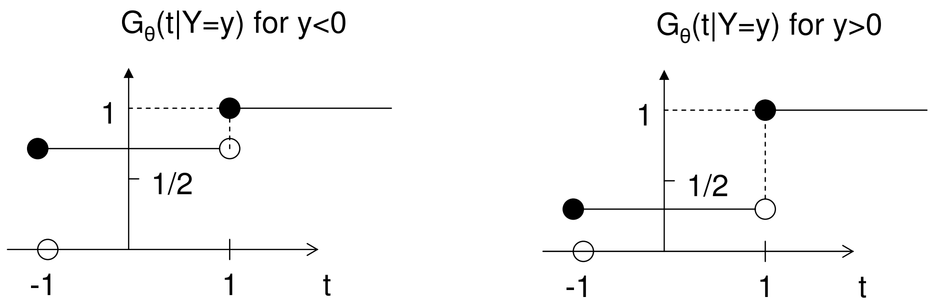

Fig. 1.

Problem 2

[2a] Here, $g(t) = 1$, $0 \le t \le 1$, and

$$P_t(Y \le y) = P(V \le y - t) = \begin{cases} 0, \ y < t \\ 1 - e^{-\alpha(y-t)}, \ y \ge t, \end{cases}$$

whence

$$f_t(y) = \alpha e^{-\alpha(y-t)} 1(y \ge t), \ \ 0 \le t \le 1.$$

Then,

$$f(t, y) = \alpha e^{-\alpha(y-t)} 1(0 \le t \le \min(y, 1)), \ y \ge 0,$$

so that

$$f(y) = \begin{cases} \int_0^y \alpha e^{-\alpha(y-t)} dt = 1 - e^{-\alpha y}, & y \le 1 \\ \int_0^1 \alpha e^{-\alpha(y-t)} dt = (e^{\alpha} - 1)\, e^{-\alpha y}, & y > 1. \end{cases}$$

Hence,

$$g(t|y) = \begin{cases} \frac{\alpha e^{-\alpha(y-t)}}{1-e^{-\alpha y}}, & 0 \le t \le y \le 1 \\ \frac{\alpha e^{-\alpha(y-t)}}{(e^{\alpha}-1)e^{-\alpha y}} = \frac{\alpha e^{\alpha t}}{e^{\alpha}-1}, & 0 \le t \le 1, y > 1. \end{cases}$$

Observe that $g(t|y)$ does not depend on y for $y > 1$ (why?)

[2b] g_{MSE} is given by

$$g_{MSE}(y) = \mathrm{E}[\theta|Y = y] = \begin{cases} \int_0^y \frac{t\alpha e^{-\alpha(y-t)}}{1-e^{-\alpha y}} dt = \frac{\alpha y}{1-e^{-\alpha y}} - 1, & 0 \leq y \leq 1 \\ \int_0^1 \frac{t\alpha e^{\alpha t}}{e^{\alpha}-1} dt = \frac{e^{\alpha}}{e^{\alpha}-1} - \frac{1}{\alpha}, & y > 1. \end{cases}$$

[2c] g_{MAP} is given by

$$g_{MAP}(y) = \begin{cases} y, & 0 \leq y \leq 1 \\ 1, & y \geq 1 \end{cases}$$

$$= \min\{y, 1\}, \ y \geq 0.$$

Problem 3

[3a] Observe that

$$\begin{aligned}\hat{\mathrm{E}}\left[X_{t+1}|X_0, X_1, \cdots, X_t\right] &= \hat{\mathrm{E}}\left[X_{t+1}|X^{t-1}, X_t\right] \\ &= \hat{\mathrm{E}}\left[X_{t+1}|X^{t-1}, X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right] \\ &= \hat{\mathrm{E}}\left[X_{t+1}|X^{t-1}\right] + \hat{\mathrm{E}}\left[X_{t+1}|X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right], \quad (1)\end{aligned}$$

since $\mathrm{E}[X_{t+1}] = 0$, and X^{t-1} and $X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]$ are uncorrelated. In (1), $X_{t+1} = (W_t + W_{t+1}) \perp X^{t-1} \in \text{span}\{W^{t-1}\}$ and $\mathrm{E}[X_{t+1}] = 0$, so that

$$\begin{aligned}\hat{\mathrm{E}}[X_{t+1}|X^t] &= \hat{\mathrm{E}}[X_{t+1}|X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]] \\ &= \alpha(t)\left(X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right)\end{aligned}$$

since $\mathrm{E}\left[X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right] = 0$.

Thus, $\hat{\mathrm{E}}[X_{t+1}|X^t] = \alpha(t)\left(X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right)$ where

$$\begin{aligned}\alpha(t) &= cov\left[X_{t+1}, X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right]\left(var[X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]]\right)^{-1} \\ &= \mathrm{E}[(W_t + W_{t+1})\left(W_{t-1} + W_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right)]\Sigma_t^{-1} \\ &= \Sigma_t^{-1},\end{aligned}$$

where $\hat{\mathrm{E}}\left[X_t|X^{t-1}\right] \in \text{span}\{W^{t-1}\}$, and $\Sigma_t = var\left[X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right]$.

[3b] The error variance in question is

$$\begin{aligned}\Sigma_t = var[X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]] &= \mathrm{E}\left[\left(X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right)^2\right] \\ &= \mathrm{E}\left[X_t\left(X_t - \hat{\mathrm{E}}\left[X_t|X^{t-1}\right]\right)\right], \quad \text{by OP} \\ &= \mathrm{E}\left[X_t\left(X_t - \left\{\alpha(t-1)\left(X_{t-1} - \hat{\mathrm{E}}\left[X_{t-1}|X^{t-2}\right]\right)\right\}\right)\right], \\ &\qquad \text{by the previous part, with } \alpha(t-1) = \Sigma_{t-1}^{-1} \\ &= \mathrm{E}[X_t^2] - \frac{1}{\Sigma_{t-1}}\mathrm{E}\left[X_t\left(X_{t-1} - \hat{\mathrm{E}}\left[X_{t-1}|X^{t-2}\right]\right)\right] \\ &= 2 - \frac{1}{\Sigma_{t-1}}.\end{aligned}$$

Note that $X_t = W_t + W_{t-1}$, $X_{t-1} = W_{t-1} + W_{t-2}$, and $\hat{\mathrm{E}}\left[X_{t-1}|X^{t-2}\right] \in \mathrm{span}\{W^{t-2}\}$.

Thus,

$$\Sigma_t = 2 - \frac{1}{\Sigma_{t-1}}, \quad t \geq 1,$$

which, with the initial condition

$$\Sigma_0 = \mathrm{E}\left[\left(X_0 - \hat{\mathrm{E}}\left[X_0|X^{-1}\right]\right)^2\right] = \mathrm{E}[X_0^2] = 2,$$

gives that

$$\Sigma_t = \frac{t+2}{t+1}, \quad t \geq 0.$$