ENEE 621: Estimation and Detection Theory

Problem Set 1: Solutions **Spring 2007**

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Problem [1] Straightforward.

Problem [2](i) Assume that the family $\{F_\theta, \theta \in \Theta\}$ satisfies the absolute continuity assumptions with respect to a distribution F on $\mathbb{R}^k$. Let $\{f_\theta, \theta \in \Theta\}$ be the corresponding family of densities. Since $\varphi_o T$ is a sufficient statistic, we know by the Factorization Theorem that there exist Borel mappings $h : \mathbb{R}^\ell \times \Theta \to [0, \infty)$ and $q : \mathbb{R}^k \to [0, \infty)$ such that for each θ in Θ, we have

$$f_\theta(y) = h(\varphi(T(y)); \theta) q(y) \quad F \text{ a.e.}$$

Consider the Borel mapping $g : \mathbb{R}^d \times \Theta \to [0, \infty)$ defined by

$$g(x; \theta) = h(\varphi(x); \theta), \quad x \in \mathbb{R}^d, \theta \in \Theta.$$

It then follows for each θ in Θ that

$$f_\theta(y) = g(T(y); \theta) q(y) \quad F \text{ a.e.},$$

so that by the Factorization Theorem, T is a sufficient statistic for $\{F_\theta, \theta \in \Theta\}$.

(ii) Since φ is invertible, we can write $T = \varphi_0^{-1}(\varphi_o T)$. Since $T = \varphi_0^{-1}(\varphi_0 T)$ is a sufficient statistic for $\{F_\theta, \theta \in \Theta\}$, then so is $\varphi_0 T$ by part (i).

Problem [3] Consider a Borel mapping $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $E_\theta[|\phi(Y)|] < \infty$ for each $\theta > 0$,

$$\sum_{y=0}^{\infty} \frac{\theta^y}{y!} \exp(-\theta) |\varphi(y)| < \infty, \ \theta > 0. \tag{1}$$

The condition $E_\theta[\varphi(Y)] = 0$ for every $\theta > 0$ yields

$$\sum_{y=0}^{\infty} \frac{\theta^y}{y!} \varphi(y) = 0, \quad \theta > 0. \tag{2}$$

We need to show that $\varphi(y) = 0$ F a.e., where F is the counting measure on $\mathbb{N}$, i.e., that $\varphi(y) = 0$ for all y in $\mathbb{N}$.

Define a mapping $\psi : C \to C$ (C = Complex plane) by

$$\psi(z) = \sum_{y=0}^{\infty} \frac{\varphi(y)}{y!} z^y, \quad z \in C.$$

This mapping is well-defined and analytic on C since

$$\sum_{y=0}^{\infty} |\frac{\varphi(y)}{y!} z^y| = \sum_{y=0}^{\infty} \frac{|\varphi(y)|}{y!} |z|^y < \infty, \ z \in C$$

in view of (1). Since (2) is equivalent to $\psi(u + i0) = 0$ for all $u > 0$, we obtain that $\psi(z) = 0$ for all z in C, by standard properties of analytic functions, i.e., $\frac{\varphi(y)}{y!} = 0$ for all y in $\mathbb{N}$, whence $\varphi(y) = 0$ for all y in $\mathbb{N}$.

Problem [4] Consider a Borel mapping $\varphi : \mathbb{R}^k \to \mathbb{R}$ satisfying $E_\theta[|\varphi(Y)|] < \infty$ for each θ in $\mathbb{R}^k$, i.e.,

$$\int_{\mathbb{R}^k} |\varphi(y)| \exp[-\frac{1}{2}(y-\theta)^T R^{-1}(y-\theta)]dy < \infty, \quad \theta \in \mathbb{R}^k .$$

The condition $E_\theta[\varphi(Y)] = 0$ for all θ in $\mathbb{R}^k$ is equivalent to

$$\int_{\mathbb{R}^k} \varphi(y) \exp[-\frac{1}{2}(y-\theta)^T R^{-1}(y-\theta)]dy = 0, \quad \theta \in \mathbb{R}^k . \tag{3}$$

We need to show as a consequence that $\phi(y) = 0$ F a.e., where F is the Lebesgue measure on $\mathbb{R}^k$. Consider the mapping $\psi : C^k \to C^k$ defined by

$$\psi(z) = \int_{\mathbb{R}^k} \varphi(y) \exp[-\frac{1}{2}(y-z)^T R^{-1}(y-z)]dy, \quad z \in C^k$$

Then, (3) means that $\psi(u + i0) = 0$ for all $u \in \mathbb{R}^k$, whence $\psi(z) = 0$ for all z in C^k. Consequently, $\varphi(y) = 0$ F a.e.

Problem [5] Pick a Borel mapping $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(y) = \begin{cases} 1, & y \geq 0 \\ -1, & y < 0. \end{cases}$$

Then $E_\theta[|\varphi(Y)|] = 1 < \infty$ for each $\theta > 0$. Observe that $E_\theta[\varphi(Y)] = 0$ for each $\theta > 0$, whereas clearly $\varphi(y) \neq 0 \quad F$ a.e. on $\mathbb{R}$. Hence, the family $\{\mathcal{N}(0,\theta), \theta > 0\}$ is not complete.

Problem [**6**] Pick a Borel mapping $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(y) = y$ for all y in $\mathbb{R}$. Then for each $\theta > 0$,

$$E_\theta[|\varphi(Y)|] = E_\theta[|Y|] = \frac{\theta}{2} < \infty.$$

Clearly, $E_\theta[\varphi(Y)] = 0$ for every $\theta > 0$, while $P_\theta[\varphi(Y) = 0] = 0$ for every $\theta > 0$. Hence, $\{\mathcal{U}(-\theta, \theta), \theta > 0\}$ is not a complete family. Note on the other hand that the (larger) family of distributions $\{\mathcal{U}(\alpha, \beta), \alpha < \beta\}$ is a complete family.

Problem [**7**] Consider the Borel mapping $\varphi : \mathbb{R}^n \to \mathbb{R}$ given by

$$\varphi(\mathbf{y}^n) = y_1 - y_2, \quad \mathbf{y}^n \in \mathbb{R}^n$$

where $n \geq 2$. Clearly, $E_\theta[\varphi(Y^n)] = E_\theta[Y_1] - E_\theta[Y_2] = 0$ for every θ in $(0,1)$, whereas

$$P_\theta[\varphi(\mathbf{y}^n) = 0] = P_\theta[Y_1 = Y_2] = \theta^2 + (1-\theta)^2 < 1, \ \theta \in (0,1),$$

the given family is not complete.

Problem [**8**] Observe that for each θ in $\mathbb{R}, \mathbf{y}^n$ is a Gaussian rv $\sim \mathcal{N}(O_n, R^{(n)}(\theta))$, with

$$(R^{(n)}(\theta))_{ij} = \theta^2 + \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Consequently, we can write the densities $\{f_\theta^{(n)}, \theta \in \mathbb{R}\}$ as

$$f_\theta^{(n)}(\mathbf{y}^n) = \frac{1}{\sqrt{(2\pi)^n \det R^{(n)}(\theta)}} \exp[-\frac{1}{2}\mathbf{y}^{n^T} R^{(n)^{-1}}(\theta)\mathbf{y}^n], \quad \mathbf{y}^n \in \mathbb{R}^{\cdot}$$

Verify that

$$R^{(n)^{-1}} = \frac{1}{(1+n\theta^2)} \begin{bmatrix} 1+(n-1)\theta^2 & -\theta^2 & \cdots & -\theta^2 \\ -\theta^2 & 1+(n-1)\theta^2 & \cdots & -\theta^2 \\ \vdots & \vdots & \ddots & \vdots \\ -\theta^2 & -\theta^2 & \cdots & 1+(n-1)\theta^2 \end{bmatrix}$$

so that

$$\begin{aligned} \mathbf{y}^{n^T} R^{(n)^{-1}}(\theta)\mathbf{y}^n &= \frac{1}{(1+n\theta^2)} \left[\sum_{i=1}^n y_i^2(1+(n-1)\theta^2) - \sum_{\substack{i \neq j \\ 1 \leq i,j \leq n}} y_i y_j \theta^2 \right] \\ &= \sum_{i=1}^n y_i^2 - \left(\frac{\theta^2}{1+n\theta^2}\right) \left[\sum_{i=1}^n y_i^2 + \sum_{\substack{u \neq j \\ 1 \leq i,j \leq n}} y_i y_j \right] \\ &= \sum_{i=1}^n y_i^2 - \left(\frac{\theta^2}{1+n\theta^2}\right) \left(\sum_{i=1}^n y_i \right)^2. \end{aligned}$$

Setting $T_n(\mathbf{y}^n) = \sum_{i=1}^n y_i$, we see that for each θ in $\mathbb{R}$,

$$f_\theta^{(n)}(\mathbf{y}^n) = h(T_n(\mathbf{y}^n);\theta)q(\mathbf{y}^n), \quad \mathbf{y}^n \in \mathbb{R}^n$$

where

$$h(T_n(\mathbf{y}^n);\theta) = \frac{1}{\sqrt{(2\pi)^n \mathrm{det} R^{(n)}(\theta)}} \exp\left[\frac{-\theta^2}{2(1+n\theta^2)}\left(T_n(\mathbf{y}^n)^2\right)\right]$$

$$q(\mathbf{y}^n) = \exp\left[\frac{1}{2}\sum_{i=1}^n y_i^2\right].$$

By the Factorization Theorem, $T_n(\mathbf{y}^n) = \sum_{i=1}^n y_i, n = 1, 2, \ldots$ is a sufficient statistic.

Problem [9](i) For each $\theta \neq 0$ in $\mathbb{R}$, we have

$$f_\theta^{(n)}(\mathbf{y}^n) = \frac{1}{\sqrt{2\pi\theta^{2n}}} \exp\left[-\frac{1}{2}\sum_{y=1}^n \frac{(y_i-\theta)^2}{\theta^2}\right], \quad \mathbf{y}^n \in \mathbb{R}^n$$

$$= \frac{e^{1/2}}{\sqrt{2\pi\theta^{2n}}} e^{\frac{1}{2\theta^2}\sum_{i=1}^n y_i^2} e^{\frac{1}{\theta}\sum_{i=1}^n y_i}, \quad \mathbf{y} \in \mathbb{R}^n,$$

So that by the Factorization Theorem,

$$T_n(\mathbf{y}^n) = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n y_i^2 \end{bmatrix}, \qquad \mathbf{y}^n \in \mathbb{R}^n$$

is a (nontrivial) sufficient statistic.

(ii) Note that

$$E_\theta\left[\left(\sum_{i=1}^n Y_i\right)^2\right] = 2n\theta^2 + n(n-1)\theta^2$$

$$= n(n+1)\theta^2$$

and

$$E_\theta\left[\sum_{i=1}^n Y_i^2\right] = 2n\theta^2.$$

Define $\phi(T_n(\mathbf{Y}^n)) = 2(\sum_{i=1}^n Y_i)^2 - (n+1)\left(\sum_{i=1}^n Y_i^2\right)$. Then, clearly

$$E_\theta\left[\varphi(T_n(\mathbf{Y}^n))\right] = 0, \quad \theta \neq 0 \text{ in } \mathbb{R}$$

while

$$P_\theta\left[\varphi(T_n(\mathbf{Y}^n)) = 0\right] \neq 1, \quad \theta \neq 0 \text{ in } \mathbb{R}$$

Hence, T_n is not a complete sufficient statistic.

Problem [10] (i) For each $\theta > 0$, we have

$$f_\theta^{(n)}(\mathbf{y}^n) = \frac{1}{\theta^n} 1(\max_{1 \le i \le n} y_i < \theta) \cdot 1(\min_{1 \le i \le n} y_i > 0),$$

so that by the Factorization Theorem,

$$T_n(\mathbf{y}^n) = \max_{1 \le i \le n} y_i, \quad \mathbf{y}^n \in \mathbb{R}^n$$

is a sufficient statistic.

We shall examine next T_n for completeness. Observer that

$$\begin{aligned} P_\theta\left[T_n \le t\right] &= \begin{cases} 0, & t \le 0 \\ (P_\theta[Y < t])^n, & 0 < t \le \theta \\ 1, & t > \theta \end{cases} \\ &= \begin{cases} 0, & t \le 0 \\ \theta^{-n} t^n, & 0 < t \le \theta \\ 1, & t > \theta. \end{cases} \end{aligned}$$

The corresponding density of T_n, denoted by h_θ, is given by

$$h_\theta(t) = \begin{cases} 0, & t \le 0 \\ n\theta^{-n} t^{n-1}, & 0 < t \le \theta \\ 0, & t > \theta. \end{cases}$$

Then, for a Borel mapping $\varphi : \mathbb{R} \to \mathbb{R}$,

$$E_\theta[\varphi(T_n)] = 0, \quad \theta > 0$$

means

$$\int_0^\theta \varphi(t) n\theta^{-n} t^{n-1} dt = 0, \quad \theta > 0$$

which implies

$$\int_0^\theta \varphi(t) t^{n-1} dt = 0, \quad \theta > 0.$$

The previous condition implies that $\varphi(t) = 0$ F a.e. on $(0, \infty)$, where F is the Lebesque measure on $\mathbb{R}$. Hence, T_n is a complete sufficient statistic.

(ii) For each $\theta > 0$, we have

$$\begin{aligned} E_\theta[T_n(\mathbf{y}^n)] &= \int_0^\theta t n\theta^{-n} t^{n-1} dt = \frac{n}{\theta^n} \int_0^\theta t^n dt \\ &= \left(\frac{n}{n+1}\right)\theta. \end{aligned}$$

Consequently, the estimator g given by

$$g(\mathbf{y}^n) = \left(\frac{n+1}{n}\right) T_n(\mathbf{y}^n), \quad \mathbf{y}^n \in \mathbb{R}^n$$

is unbiased. Clearly, g is also a MVUE.

Problem [11] For each $\theta = \alpha^2 > 0, Y \sim \mathcal{N}(0, 1+\theta)$. The family of distributions $\{\mathcal{N}(0, 1+\theta), \theta > 0\}$ is not complete. To see this, pick a Borel mapping $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(y) = \begin{cases} 1, & y \geq 0 \\ -1, & y < 0. \end{cases}$$

Then, $E_\theta[|\phi(Y)|] = 1 < \infty$ for each $\theta > 0$. While $E_\theta[\varphi(Y)] = 0$ for each $\theta > 0$, clearly $\varphi(y) \neq 0$ F a.e.