ENEE 621: Estimation and Detection Theory

Problem Set 1 **Spring 2007**

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1 Let $T : \mathbb{R}^k \to \mathbb{R}^d$ be a sufficient statistic for the family of probability distributions $\{F_\theta, \theta \in \Theta\}$ on $\mathbb{R}^k$. Thus, for each θ in Θ,

$$\mathbf{P}_\theta \left[Y \in B | T(Y)\right] = \nu(B, T(Y)), \quad B \in \mathcal{B}(\mathbb{R}^k) \qquad \mathbf{P}_\theta - a.s.,$$

for some mapping $\nu : \mathcal{B}(\mathbb{R}^k) \times \mathbb{R}^d \to [0,1]$. Consider the Bayesian situation in which θ is a Θ-valued rv with probability distribution G. With $\mathbf{P}$ being the probability measure on $\Theta \times \mathbb{R}^k$ as defined in class, show that

$$\mathbf{P} \left[Y \in B | T(Y), \theta\right] = \nu'(B, T(Y)), \quad B \in \mathcal{B}(\mathbb{R}^k) \quad \mathbf{P} - a.s.,$$

for some mapping $\nu' : \mathcal{B}(\mathbb{R}^k) \times \mathbb{R}^d \to [0,1]$.

2. Consider a family of probability distributions $\{F_\theta, \theta \in \Theta\}$ on $\mathbb{R}^k$. For a statistic $T : \mathbb{R}^k \to \mathbb{R}^d$ and a Borel mapping $\varphi : \mathbb{R}^d \to \mathbb{R}^\ell$, show that:

(i) if $\varphi_o T$ is a sufficient statistic for $\{F_\theta, \theta \in \Theta\}$, then so is T;

(ii) if T is a sufficient statistic for $\{F_\theta, \theta \in \Theta\}$, then so is $\varphi_o T$ provided the mapping φ is invertible.

3. Show that the family of Poisson distributions $\{\mathcal{P}(\theta), \theta > 0\}$ is a complete family.

4. Show that the family of Gaussian distributions $\{\mathcal{N}(\theta, R), \theta \in \mathbb{R}^k\}$ is a complete family.

5. Consider the family of Gaussian distributions $\{\mathcal{N}(0, \theta), \theta > 0\}$ on $\mathbb{R}$. Determine whether this family is complete.

6. Consider the family of uniform distributions $\{\mathcal{U}(-\theta, \theta), \theta > 0\}$. Is this family complete?

7. Consider the family of distributions $\{Ber^{(n)}(\theta), \theta \in (0,1)\}$ corresponding to n i.i.d. observations of a Bernoulli rv. Show that this family is not complete if $n \geq 2$.

8. Consider an estimation problem in which the observations $\{Y_n, n = 1, 2, \ldots\}$ are $\mathbb{R}$-valued rvs given by

$$Y_n = \theta X + V_n, \quad n = 1, 2, \ldots$$

where θ is in $\mathbb{R}$, and the rvs $\{X, V_n, n = 1, 2, \ldots\}$ are i.i.d. $\mathcal{N}(0, 1)$. Find a sufficient statistic for θ (on the basis of Y^n).

9. For each $\theta \neq 0$ in $\mathbb{R}$, let $(Y_1, \ldots, Y_n)$, $n = 1, 2, \ldots$, be i.i.d. $\mathbb{R}$-valued $\sim \mathcal{N}(\theta, \theta^2)$ rv's.

(i) Find a nontrivial sufficient statistic for estimating the parameter (θ, θ^2) on the basis of $Y^n = (Y_1, \ldots, Y_n)$.

(ii) Determine whether the statistic in part (i) above is a complete sufficient statistic.

10. We are interested in estimating a parameter $\theta > 0$ on the basis of n i.i.d. observations, $Y_1, \ldots, Y_n, n = 1, 2, \ldots$, each of which is uniformly distributed on the interval $(0, \theta)$.

(i) For the associated family of distributions $\{\mathcal{U}^{(n)}(0, \theta), \theta > 0\}$, find a nontrivial sufficient statistic which is also complete.

(ii) Determine a MVUE for θ on the basis of $Y^n = (Y_1, \ldots, Y_n)$.

11. Consider the $\mathbb{R}$-valued observation rv Y given by

$$Y = \alpha X + N, \quad \alpha \neq 0$$

where X and N are independent Gaussian rvs, each with mean zero and unit variance, and α is a nonzero scalar. The parameter to be estimated is $\theta = \alpha^2$. Determine whether the family of distributions $\{F_\theta, \theta > 0\}$ of the rv Y is complete.