ENEE 621: Estimation and Detection Theory

Problem Set 2: Solutions **Spring 2007**

Narayan

Problem 1. First observe that the family $\{\mathcal{U}(1,\ldots,\theta), \theta \in \{1,2,\ldots\}\}$ is complete. For any $\varphi : \mathbb{R} \to \mathbb{R}$ with $E_\theta[|\varphi(Y)|] = \sum_{y=1}^{\theta} |\varphi(y)|\frac{1}{\theta} < \infty, \theta \in \{1,2,\ldots\}$, we have that $E_\theta[\varphi(Y)] = 0, \theta \in \{1,2,\ldots\}$, implies $\sum_{y=1}^{\theta} \varphi(y) = 0, \theta \in \{1,2,\ldots\}$, i.e., $\varphi(y) = 0, y \in \{1,2\ldots\}$. Consequently, if $g : \mathbb{R} \to \mathbb{R}$ is an unbiased estimator of θ on the basis of Y, it is "essentially unique": if $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is another unbiased estimator, then it must hold by the completeness of $\{\mathcal{U}(1,\ldots,\theta), \theta \in \{1,2\ldots\}\}$ that $\tilde{g}(Y) = g(Y)$ P_θ-a.s. $\theta \in \{1,2,\ldots\}$, whence $\tilde{g}(y) = g(y), y \in \{1,2,\ldots\}$.

Next, $E_\theta[g(Y)] = \theta, \theta \in \{1,2,\ldots\}$, implies that

$$\sum_{y=1}^{\theta} g(y) = \theta^2, \ \theta \in \{1,2,\ldots\}. \tag{1}$$

Consequently, $g(1) = 1, g(2) = 3, g(3) = 5, \ldots$, suggesting that a solution to (1) is given by $g(y) = 2y-1, y \in \{1,2,\ldots\}$. This is, in fact, true and can be verified by direct substitution in (1) since $\sum_{y=1}^{\theta}(2y-1) = \theta^2$.

Thus, the estimator $g : \mathbb{R} \to \mathbb{R}$ satisfying $g(y) = 2y-1, y \in \{1,2,\ldots\}$, is the only unbiased estimator for θ on the basis of Y.

Problem 2. (a) Since the family $\{\mathbf{G}(\theta), \theta \in (0,1)\}$ is complete, if $g : \mathbb{R} \to \mathbb{R}$ is an unbiased estimator of θ on the basis of Y, then g is "essentially unique."

Next, $E_\theta[g(Y)] = \theta, \quad \theta \in (0,1)$ implies

$$\sum_{y=0}^{\infty} g(y)(1-\theta)^y = 1, \quad \theta \in (0,1). \tag{2}$$

Equating coefficients of powers of $(1-\theta)$, we see that

$$g(y) = \begin{cases} 1, & y = 0 \\ 0, & y = 1,2,\ldots \end{cases}$$

Also,

$$\begin{aligned}\Sigma_\theta(g) &= E_\theta[(g(Y)-\theta)^2] = E_\theta[g^2(Y)] - \theta^2 \\ &= P_\theta[g(Y)=1] - \theta^2 = P_\theta[Y=0] - \theta^2 = \theta(1-\theta), \theta \in (0,1).\end{aligned}$$

(b)

$$E_\theta[\tilde{g}(Y)] = \sum_{y=1}^{\infty} \frac{1}{y}\theta(1-\theta)^y = \theta \sum_{y=1}^{\infty} \frac{(1-\theta)^y}{y} = -\theta \ln\theta, \quad \theta \in (0,1), \tag{3}$$

so that $\tilde{g}$ is a biased estimator. Next,

$$\begin{aligned}\Sigma_\theta(\tilde{g}) &= E_\theta[(\tilde{g}(Y)-\theta)^2] = E_\theta[\tilde{g}^2(Y)] + \theta^2 - 2\theta(-\theta\ln\theta) \\ &= \sum_{y=1}^{\infty} \frac{1}{y^2}\theta(1-\theta)^y + \theta^2(1+2\ln\theta) \\ &< \sum_{y=1}^{\infty} \frac{1}{y}\theta(1-\theta)^y + \theta^2(1+2\ln\theta) \\ &= -\theta\ln\theta + \theta^2(1+2\ln\theta), \quad \text{by (3)} \\ &= \theta^2 + \theta(2\theta-1)\ln\theta, \\ &= \theta[\theta + (2\theta-1)\ln\theta],\end{aligned}$$

i.e.,

$$\Sigma_\theta(\tilde{g}) < \theta[\theta + (2\theta-1)\ln\theta], \ \theta \in (0,1). \tag{4}$$

It can now be directly verified that

$$\theta[\theta + (2\theta-1)\ln\theta] < \Sigma_\theta(g) = \theta(1-\theta), \quad \theta \in (e^{-1}, 1),$$

so that (at least) for $\theta \in (e^{-1}, 1)$, we have

$$\Sigma_\theta(\tilde{g}) < \Sigma_\theta(g).$$

(c) Verify that (CR1)-(CR4) are satisfied. Now, for each $\theta \in (0,1)$,

$$\ln(f_\theta(y)) = \ln\theta + y\ln(1-\theta), y \in \mathbb{N}.$$

For each θ in (0,1), we then have

$$\begin{aligned} M(\theta) &= E_\theta\left[|\frac{d}{d\theta}\ln f_\theta(Y)|^2\right] \\ &= E_\theta\left[(\frac{1}{\theta} - \frac{Y}{1-\theta})^2\right] = E_\theta\left[\frac{1}{(1-\theta)^2}(Y - \frac{1-\theta}{\theta})^2\right] \\ &= \frac{1}{(1-\theta)^2}\frac{(1-\theta)}{\theta^2} \\ &= \frac{1}{\theta^2(1-\theta)}. \end{aligned}$$

(d) Since $\Sigma_\theta(g) = \theta(1-\theta)$ and $M^{-1}(\theta) = \theta^2(1-\theta), \theta \in (0,1)$, we see that $\Sigma_\theta(g) > M^{-1}(\theta), \theta \in (0,1)$, so that g is not an efficient estimator.

Problem 3. The family $\{\mathcal{N}^{(n)}(\theta, \sigma^2), \theta \in \mathbb{R}\}$ is an exponential family, with the correspondence $K^{(n)}(\mathbf{y}^n) = \sum_{i=1}^n y_i, \mathbf{y}^n = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Furthermore, $K^{(n)}$ is a complete sufficient statistic for the family. Next, since $E_\theta\left[K^{(n)}(\mathbf{Y}^n)\right] = E_\theta\left[\sum_{i=1}^n Y_i\right] = n\theta, \theta \in \Theta$, it is clear that

$$T_n(\mathbf{Y}^n) \triangleq \frac{1}{n}\sum_{i=1}^n Y_i$$

is an unbiased estimator for θ on the basis of $\mathbf{Y}^n$, which is also a complete sufficient statistic. Thus, the three-step method simplifies in this case to yield that the estimator $g : \mathbb{R}^n \to \mathbb{R}$ given by $g(\mathbf{y}^n) = \frac{1}{n}\sum_{i=1}^n y_i$ is a MVUE.

Problem 4. (a) For each θ in (0,1), we have

$$f_\theta^{(n)}(\mathbf{y}^n) = \Pi_{i=1}^n f_\theta(y_i) = \theta^n(1-\theta)^{\sum_{i=1}^n y_i}, \ \mathbf{y}^n \in \mathbb{N}^n,$$

so that $T_n(\mathbf{y}^n) \triangleq \sum_{i=1}^n y_i$ defines a sufficient statistic for the family $\{\mathcal{G}^{(n)}(\theta), \theta \in (0,1)\}$.

(b) The family $\{\mathcal{G}^{(n)}(\theta), \theta \in (0,1)\}$ is an exponential family with the correspondence $K^{(n)}(\mathbf{y}^n) = T_n(\mathbf{y}^n) = \sum_{i=1}^n y_i$, so that T_n is a completely sufficient statistic for the family (since $K^{(n)}$ is a complete sufficient statistic).

(c) For each θ in (0,1), the Fisher information matrix $M^{(n)}(\theta), n = 1, 2, \ldots$, is given by

$$\begin{aligned} M^{(n)}(\theta) &= nM(\theta) \\ &= \frac{n}{\theta^2(1-\theta)}. \end{aligned}$$

(d) Clearly, there exist no efficient estimators for θ on the basis of $Y^n, n = 1, 2, \ldots$, as is clear from the answer to Problem 2, part (d), for the case $n = 1$.

Problem 5. (a) For each $\theta \in \mathbb{N}$, we have

$$f_\theta(\mathbf{y}^n) = \frac{1}{\theta^n} 1\left(\max_{1 \le i \le n} y_i \le \theta\right) 1\left(\min_{1 \le i \le n} y_i \ge 1\right), \quad \mathbf{y}^n \in \{1, \ldots, \theta\}^n,$$

so that by the Factorization Theorem,

$$T_n(\mathbf{y}^n) \triangleq \max_{1 \le i \le n} y_i, \quad \mathbf{y}^n \in \mathbb{N}^n$$

is a sufficient statistic.

(b) For $t = 1, \ldots, \theta$, note that

$$P_\theta[T_n \le t] = P_\theta\left[\max_{1 \le i \le n} Y_i \le t\right] = \left(\frac{t}{\theta}\right)^n$$

so that

$$P_\theta[T_n = t] = \begin{cases} \left(\frac{t}{\theta}\right)^n - \left(\frac{t-1}{\theta}\right)^n, & t = 1, \ldots, \theta \\ 0, & \text{elsewhere.} \end{cases}$$

For any Borel mapping $\varphi : \mathbb{R} \to \mathbb{R}$ with $E_\theta[|\varphi(T_n)|] < \infty$, we have

$$E_\theta[\varphi(T_n)] = 0, \quad \theta \in \mathbb{N}$$

means

$$\sum_{t=1}^{\theta} \varphi(t) \cdot \frac{1}{\theta^n}[t^n - (t-1)^n] = 0, \quad \theta \in \mathbb{N}$$

so that

$$\sum_{t=1}^{\theta} \varphi(t)[t^n - (t-1)^n] = 0, \quad \theta \in \mathbb{N}\,.$$

By examining the previous condition for $\theta = 1, 2, \ldots$, it is readily seen that

$$\varphi(t) = 0, \quad t \in \mathbb{N},$$

so that T_n is a complete sufficient statistic.

(c),(d): The condition CR2(a) is not satisfied, so that $M(\theta)$ and efficiency are not defined.

Problem 6. (a) For each $\theta > 0$, we have

$$f_\theta^{(n)}(\mathbf{y}^n) = \frac{1}{(2\pi\theta)^{n/2}} \exp\left[-\frac{1}{2\theta}\sum_{i=1}^{n} y_i^2\right], \quad \mathbf{y}^n \in \mathbb{R}^n$$

so that $T_n(\mathbf{y}^n) \triangleq \sum_{i=1}^n y_i^2$, $\mathbf{y}^n \in \mathbb{R}^n$, is a sufficient statistic by the Factorization theorem.

(b) The family $\{\mathcal{N}^{(n)}(0,\theta), \theta > 0\}$ is an exponential family with the correspondence $K^{(n)}(\mathbf{y}^n) = T_n(\mathbf{y}^n) = \sum_{i=1}^{n} y_i^2$, so that the fact that $K^{(n)}$ is a complete sufficient statistic for this family implies the same for T_n.

(c) Verify first that the family $\{\mathcal{N}(0,\theta), \theta > 0\}$ satisfies (CR1)-(CR4) and the additional integrability and regularity conditions, so that for each $\theta > 0$,

$$\begin{aligned} M(\theta) &= -E_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(Y)\right] \\ &= -E_\theta\left[\frac{1}{2\theta^2} - \frac{Y^2}{\theta^3}\right] = -\frac{1}{2\theta^2} + \frac{\theta}{\theta^3} \\ &= \frac{1}{2\theta^2}. \end{aligned}$$

(d) Since $E_\theta\left[T_n(Y^n)\right] = E_\theta\left[\sum_{i=1}^n Y_i^2\right] = n\theta, \theta > 0$, it is clear that the estimator $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(y^n) \triangleq \frac{1}{n}\sum_{i=1}^{n} y_i^2, \mathbf{y}^n \in \mathbb{R}^n$, is an unbiased estimator of θ on the basis of Y^n. Furthermore, since the statistic $T_n : \mathbb{R}^n \to \mathbb{R}$ given by $T_n(\mathbf{y}^n) = \frac{1}{n}\sum_{i=1}^{n} y_i^2, \mathbf{y}^n \in \mathbb{R}^n$, is a complete sufficient statistic (by part (b)), it follows that g is a

MVUE. To see if g is also efficient, we see that for each $\theta > 0$,

$$\begin{aligned}
\Sigma_\theta(g) &= E_\theta\left[|g(\mathbf{Y}^n) - \theta|^2\right] = E_\theta\left[\left(\frac{1}{n}\sum_{i=1}^n Y_i^2 - \theta\right)^2\right] \\
&= E_\theta\left[\left\{\frac{1}{n}\sum_{i=1}^n (Y_i^2 - \theta)\right\}^2\right] \\
&= \frac{1}{n^2} E_\theta\left[\sum_{i=1}^n \left(Y_i^2 - \theta\right)^2 + \sum_{\substack{i,j=1 \\ i\neq j}}^n \left(Y_i^2 - \theta\right)\left(Y_j^2 - \theta\right)\right] \\
&= \frac{1}{n^2} E_\theta\left[\sum_{i=1}^n (Y_i^2 - \theta)^2\right], \text{ by the independence of } Y_1, \ldots, Y_n \\
&= \frac{1}{n}\left[E_\theta\left[Y^4\right] + \theta^2 - 2\theta^2\right] = \frac{1}{n}\left[E_\theta\left[Y^4\right] - \theta^2\right] \\
&= \frac{1}{n}\left[3\theta^2 - \theta^2\right] = \frac{2\theta^2}{n}.
\end{aligned}$$

Since $\Sigma_\theta(g) = M^{(n)^{-1}}(\theta), \theta > 0$, we see that g is an efficient estimator.

Problem 7. (a) For each $\theta > 0$, we have

$$\begin{aligned}
f_\theta^{(n)}(\mathbf{y}^n) &= \Pi_{i=1}^n\left[\theta e^{-\theta y_i} 1(y_i \geq 0)\right] \\
&= \theta^n e^{-\theta \sum_{i=1}^n y_i} 1\left(\min_{1\leq i\leq n} y_i \geq 0\right)
\end{aligned}$$

so that by the Factorization theorem, we have that the statistic T_n defined by $T_n(\mathbf{y}^n) = \sum_{i=1}^n y_i$ is a sufficient statistic for $\{\mathcal{E}^{(n)}(\theta), \theta > 0\}$.

(b) The family $\{\mathcal{E}^{(n)}(\theta), \theta > 0\}$ constitutes an exponential family with the correspondence $K^{(n)}(\mathbf{y}^n) = T_n(\mathbf{y}^n) = \sum_{i=1}^n y_i,\ \mathbf{y}^n \in \mathbb{R}^n$. Since $K^{(n)}$ is a complete sufficient statistic for this family, so is T_n.

(c) Verify that the family $\{\mathcal{E}(\theta), \theta > 0\}$ satisfies (CR1)-(CR4). Then, for each $\theta > 0$,

$$\begin{aligned}
M(\theta) &= E_\theta\left[|\frac{d}{d\theta}\ln f_\theta(Y)|^2\right] = E_\theta\left[\left(Y - \frac{1}{\theta}\right)^2\right] \\
&= \mathrm{var}_\theta[Y] = \frac{1}{\theta^2},
\end{aligned}$$

so that

$$\begin{aligned}
M^{(n)}(\theta) &= nM(\theta) \\
&= \frac{n}{\theta^2},\ n = 1, 2, \ldots
\end{aligned}$$

Problem 8. Let $T_c : \mathbb{R}^k \to \mathbb{R}^d$ be a complete sufficient statistic for $\{F_\theta, \theta \in \Theta\}$. Let $T : \mathbb{R}^k \to \mathbb{R}^\ell$ be a sufficient statistic for $\{F_\theta, \theta \in \Theta\}$, where ℓ is a positive integer. Define a Borel mapping $\varphi_T : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\varphi_T(T_c(Y)) = T_c(Y) - E_\theta[T_c(Y)|T(Y)].$$

The sufficiency of T ensures that the right side above does not depend on θ, so that the mapping φ_T is well-defined. Now, for each $\theta \in \Theta$,

$$E_\theta\left[\varphi_T(T_c)\right] = E_\theta[T_c(Y)] - E_\theta[E_\theta(T_c(Y)|T(Y)] = 0,$$

by standard properties of conditional expectation. Hence, by the completeness of T_c, we have that $\varphi_T(T_c) = 0$ P_θ-a.s., for all $\theta \in \Theta$, i.e.,

$$T_c(Y) = E_\theta\left[T_c(Y)|T(Y)\right] \; P_\theta - \text{a.s.}, \theta \in \Theta.$$

Thus, for each $\theta \in \Theta$, we see that $T_c(Y)$ is equal to a function of $T(Y)$ P_θ-a.s. Since the sufficient statistic T was chosen arbitrarily, and since T_c is a sufficient statistic, it follows that T_c is a minimal sufficient statistic for $\{F_\theta, \theta \in \Theta\}$.

Problem 9. (a) Verify first that the family $\{F_{\tilde\theta}, \tilde\theta \in (0,1)\}$ satisfies (CR1)-(CR4). In particular, the densities $\{f_{\tilde\theta}(y) = P_{\tilde\theta}[Y = y]\}$ are given by

$$f_{\tilde\theta}(y) = P_{\tilde\theta}\left[Y = y\right] = \frac{\left(\ln \frac{1}{\tilde\theta}\right)^y \tilde\theta}{y!}, \quad y \in \mathbb{N},$$

for each $\tilde\theta \in (0,1)$. Hence,

$$\ln f_{\tilde\theta}(y) = \ln \tilde\theta + y \ln \ln \frac{1}{\tilde\theta} - \ln y!, \;\; y \in \mathbb{N}$$

so that

$$\frac{d}{d\tilde\theta} \ln f_{\tilde\theta}(y) = \frac{1}{\tilde\theta} + Y \frac{1}{\tilde\theta \ln \tilde\theta}, \quad y \in \mathbb{N}\,.$$

Then, for each $\tilde{\theta} \in (0,1)$, we have

$$\begin{aligned} M(\tilde{\theta}) &= E_{\tilde{\theta}}\left[|\frac{d}{d\tilde{\theta}} \ln f_{\tilde{\theta}}(Y)|^2 \right] \\ &= \sum_{y=0}^{\infty} \left(\frac{1}{\tilde{\theta}} + y \frac{1}{\tilde{\theta} \ln \tilde{\theta}} \right)^2 \frac{\tilde{\theta} \left(\ln \frac{1}{\tilde{\theta}} \right)^y}{y!} \\ &= \sum_{y=0}^{\infty} \left(e^{\theta} - y \frac{1}{\theta e^{-\theta}} \right)^2 \frac{e^{-\theta} \theta^y}{y!} \\ &= e^{2\theta} - \frac{2e^{2\theta}\theta}{\theta} + \frac{e^{2\theta}}{\theta^2}[\theta + \theta^2] \\ &= \frac{e^{2\theta}}{\theta} \\ &= \frac{1}{\tilde{\theta}^2} \ln \frac{1}{\tilde{\theta}}, \end{aligned}$$

i.e.,

$$M(\tilde{\theta}) = \frac{1}{\tilde{\theta}^2} \ln \frac{1}{\tilde{\theta}}, \ \tilde{\theta} \in (0,1).$$

(b) If g is an unbiased estimator of $\tilde{\theta}$ on the basis of Y, then for each $\tilde{\theta}$ in (0,1),

$$E_{\tilde{\theta}}[g(Y)] = \tilde{\theta}$$

or,

$$\sum_{y=0}^{\infty} g(y) \tilde{\theta} \frac{\left(\ln \left(1/\tilde{\theta} \right) \right)^y}{y!} = \tilde{\theta}$$

whence

$$\sum_{y=0}^{\infty} \frac{g(y)}{y!} \theta^y = 1, \ \theta \in (0,\infty).$$

By equating coefficients of the powers of θ, we obtain that

$$\frac{g(y)}{y!} = \begin{cases} 1, & y = 0 \\ 0, & \text{else} \end{cases}$$

whence

$$g(y) = \begin{cases} 1, & y = 0 \\ 0, & \text{else,} \end{cases}$$

and this defines the only unbiased estimator of $\tilde{\theta}$ on the basis of Y.

(c) For each $\tilde{\theta} \in (0,1)$, we have

$$\begin{aligned}\Sigma_{\tilde{\theta}}(g) &= E_{\tilde{\theta}}\left[|g(Y) - \tilde{\theta}|^2\right] = E_{\tilde{\theta}}\left[g^2(Y)\right] - \tilde{\theta}^2 \\ &= P_{\tilde{\theta}}\left[g(Y) = 1\right] - \tilde{\theta}^2 = P_{\tilde{\theta}}\left[Y = 0\right] - \tilde{\theta}^2 \\ &= \tilde{\theta} - \tilde{\theta}^2 = \tilde{\theta}(1 - \tilde{\theta}).\end{aligned}$$

Finally, observe that $\sum_{\tilde{\theta}}(g) = \tilde{\theta}(1-\tilde{\theta}) = e^{-\theta}(1-e^{-\theta})$, where $\theta = \ln\frac{1}{\tilde{\theta}}$. From 6 in part (a), we see that $M^{-1}(\tilde{\theta}) = \theta e^{-2\theta}$, where $\theta = \ln\frac{1}{\tilde{\theta}}$. Since $e^{-\theta}(1 - e^{-\theta}) > \theta e^{-2\theta}, \theta > 0$, we see that $\sum_{\tilde{\theta}}(g) > M^{-1}(\tilde{\theta}), \tilde{\theta} \in (0,1)$. Clearly, g is not an efficient estimator of $\tilde{\theta}$ on the basis of Y.

Problem 10. (a) It can be verified that (CRl)-(CR4) and the additional regularity conditions hold. For each θ in (0,1),

$$\ln f_\theta(y) = \ln\binom{m}{y} + y\ln\theta + (m-y)\ln(1-\theta), \quad y = 0, 1, \ldots, m,$$

whence

$$\frac{d^2}{d\theta^2}\ln f_\theta(y) = -\frac{y}{\theta^2} - \frac{m-y}{(1-\theta)^2}.$$

Hence,

$$\begin{aligned}M(\theta) &= -E_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(Y)\right] = E_\theta\left[\frac{Y}{\theta^2}\right] + E_\theta\left[\frac{m-Y}{(1-\theta)^2}\right] \\ &= \frac{m\theta}{\theta^2} + \frac{m - m\theta}{(1-\theta)^2} = \frac{m}{\theta(1-\theta)}, \quad \theta \in (0,1).\end{aligned}$$

(b) Since $E_\theta[Y] = m\theta$, the estimator g given by $g(y) = \frac{y}{m}, y \in \mathbb{R}$ is unbiased. Further, for each θ in (0,1),

$$\begin{aligned}\Sigma_\theta(g) &= E_\theta\left[\left(\frac{Y}{m} - \theta\right)^2\right] = \frac{1}{m^2}E\left[(Y - m\theta)^2\right] = \frac{1}{m^2}\operatorname{cov}_\theta[Y] \\ &= \frac{\theta(1-\theta)}{m} = M^{-1}(\theta), \text{ so that } g \text{ is efficient.}\end{aligned}$$