ENEE 621: Estimation and Detection Theory

Problem Set 2 **Spring 2007**
Narayan

1. Find all the unbiased estimators for estimating the parameter θ from the parameter set $\Theta = \{0, 1, 2, \ldots\}$, given that the observation rv Y is uniformly distributed on $\{0, \ldots, \theta\}$.

2. Let $\Theta = (0, 1)$. For each θ in Θ, let the rv Y be a geometric rv with parameter θ, i.e.,

$$\mathbf{P}_\theta[Y = y] = \theta(1-\theta)^y, \quad y = 0, 1, \ldots$$

(a) Find all the unbiased estimators of θ on the basis of Y.

(b) Consider the estimator $\tilde{g} : \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{g}(y) = \begin{cases} \frac{1}{y}, & y \neq 0 \\ 0, & y = 0. \end{cases}$$

Show that $\tilde{g}$ is a biased estimator which yields a smaller error covariance than *any* unbiased estimator, at least for certain values of θ.

(c) Determine the Fisher information matrix $M(\theta)$ for each θ in Θ.

(d) Determine if the set of estimators found in part (a) contains an efficient estimator.

3. Let $\Theta = \mathbb{R}$. Let $Y^n = (Y_1, \ldots, Y_n)$ consist of n i.i.d. Gaussian rv's, each with unknown mean θ and know variance σ^2. Find a MVUE for θ on the basis of Y^n, using the three-step method described in the class.

4. Let $\{\mathcal{G}(\theta), \theta \in (0, 1)\}$ denote the family of geometric distributions in Problem 2. Consider the families of distributions $\{\mathcal{G}^{(n)}(\theta), \theta \in (0, 1)\}$, $n = 1, 2, \ldots$, associated with n i.i.d. observations $Y^n = (Y_1, \ldots, Y_n)$.

(a) Find (nontrivial) sufficient statistics for the family $\{\mathcal{G}^{(n)}(\theta), \theta \in (0, 1)\}$.

(b) Are the statistics in part (a) above complete sufficient statistics?

(c) Compute the Fisher information matrix $M^{(n)}(\theta), \theta \in (0, 1)$.

(d) Determine the efficient estimators for θ on the basis of Y^n.

5. Repeat Problem 4 for the family of discrete uniform distributions $\{\mathcal{U}(1, \ldots, \theta), \theta \in \mathbb{N}\}$.

6. Repeat Problem 4 for the family of Gaussian distributions $\{\mathcal{N}(0,\theta), \theta > 0\}$.

7. Repeat Problem 4 for the family of exponential distributions $\{\mathcal{E}(\theta), \theta > 0\}$.

8. Let $T_c : \mathbb{R}^k \to \mathbb{R}^d$ be a complete sufficient statistic for the family $\{F_\theta, \theta \in \Theta\}$. Show that T_c is then a minimal sufficient statistic for this family.

9. This problem concerns the direct estimation of a function $d(\theta)$ of the parameter θ, rather than θ itself.

Let $\Theta = (0,\infty)$, and for $\theta \in \Theta$, let Y be a Poisson rv with parameter θ, i.e.,

$$\mathbf{P}_\theta[Y = y] = \frac{\theta^y e^{-\theta}}{y!}, \quad y \in \mathbb{N}.$$

Define a mapping $d : (0,\infty) \to (0,1)$ by

$$d(\theta) = e^{-\theta}$$

and denote $\tilde{\theta} = d(\theta)$.

(a) For each $\tilde{\theta}$ in $\tilde{\Theta} = (0,1)$, compute the Fisher information matrix $M(\tilde{\theta})$.

(b) Find all the unbiased estimators of $\tilde{\theta}$ on the basis of a (single) observation Y.

(c) For each estimator g in part (b) above, compute the error covariance $\sum_{\tilde{\theta}}(g)$.

10. For each θ in $(0,1)$, let the rv Y be a binomial rv, with

$$P_\theta[Y = y] = \binom{m}{y} \theta^y (1-\theta)^{m-y}, \qquad y = 0,1,\ldots,m,$$

where m is a fixed positive integer, and $\binom{m}{y} = \frac{m!}{y!(m-y)!}, y = 0,1,\ldots,m$.

(a) Compute the Fisher information matrix $M(\theta), \theta \in (0,1)$.

(b) Determine an efficient estimator for θ on the basis of Y.