ENEE 621: Estimation and Detection Theory

Problem Set 3 **Spring 2007**

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1. A $\mathbb{R}$-valued r.v. $\theta \sim \mathcal{N}(m, \sigma^2)$ is observed through a digital instrument with quantization levels of width q. A reasonable approach for "small" q is to model the quantizer as a noise source Z whose distribution is uniform over $\left[-\frac{q}{2}, \frac{q}{2}\right]$ with Z being independent of θ. The observation r.v. Y is given by

$$Y = \theta + Z.$$

   (i) Find the linear mean-square error (LMSE) estimate of θ given $Y = y$.

   (ii) Find the MLE of θ given $Y = y$, assuming now that θ **is an unknown $\mathbb{R}$-valued constant**.

2. The R.V. Y is uniformly distributed on $(0, \theta]$.

   (i) Let θ be an unknown constant in $(0, 1)$. Find the MLE of θ given $Y = y$.

   (ii) Now let θ be a $(0, 1)$-valued r.v. with probability density function $g_\theta(\cdot)$ given by.

$$g_\theta(t) = \begin{cases} 0, t < 0 \\ 2t, 0 \leq t < 1 \\ 0, t \geq 1. \end{cases}$$

   (a) Find the MAP estimate of θ given $Y = y$.

   (b) Find the minimum mean-squared error estimate (MMSE) of θ given $Y = y$.

   (c) Find the LMSE estimate of θ given $Y = y$.

   In each case, check the estimate for bias.

3. (Poor, p. 200, #15). Let Y be a Poisson r.v. with unknown rate θ in $(0, \infty)$. Find the MLE of θ on the basis of $Y = y$, and compute its bias and error covariance. Also, determine the corresponding Cramér-Rao bound.

4. (Poor, p. 201, #17). Let Y_1 and Y_2 be $\mathbb{R}$-valued jointly Gaussian r.v.'s, each with zero mean and unit variance. Find the MLE of the correlation coefficient $\theta = E[Y_1 Y_2]$, lying in (-1,1), on the basis of $(Y_1 Y_2) = (y_1, y_2)$. Compute the Cramér-Rao lower bound.

5. (Poor, p. 203, #23). Let $Y_k = A \sin\left(\frac{k\pi}{2} + \Phi\right) + N_k$, $k = 1, \ldots, n$, where $\{N_k\}_{k=1}^n$ are i.i.d. $\sim \mathcal{N}(0, \sigma^2)$ r.v.'s, with n being even. If $\theta = (A, \Phi)$ with $A > 0$, Φ in $(-\pi, \pi)$, find the MLE of θ on the basis of $Y = y$.

6. Let $Y_1, \ldots, Y_n$ be $\mathbb{R}$-valued i.i.d. r.v.'s with (common) density function

$$f_\theta(y) = \exp\{-(y-\theta)\}u(y-\theta), \quad \theta \in \mathbb{R}$$

where $u(\cdot)$ is the unit step function. Find the MLE of θ given $(y_1, \ldots, y_n)$.

7. Consider the lognormal density

$$f_\theta(y) = \begin{cases} (\sqrt{2\pi} y\sigma)^{-1} \exp\left\{-\frac{1}{2}\left|\frac{\ln(y/\theta)}{\sigma}\right|^2\right\}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\sigma > 0$, and θ is an unknown parameter in $(0, \infty)$. find the MLE of θ on the basis of the i.i.d. observations $\{Y_k\}_{k=1}^n$, each with common density f_θ. Is $g_{ML,n}$ a consistent estimator? Is it efficient? Compute its bias.

[**HINT**: Useful fact: $Y \sim$ lognormal with parameters $(\theta, \sigma^2) \Leftrightarrow Y = \exp Z$, where $Z \sim \mathcal{N}(\ln\theta, \sigma^2)$.]

8. Consider a discrete-time Markov chain $\{Y_k\}_0^\infty$ with states $\{1, \ldots, M\}$, $M > 1$, and stationary transition probabilities

$$\theta_{ij} \triangleq P\left(Y_{k+1} = j | Y_k = i\right), \quad 1 \le i, j \le M$$

for $k \ge 0$. (Thus, Y_k denotes the state at time instant k.) We assume that the initial state $Y_0 = y_0$ is **fixed**. If the chain is observed for n time instants, find the MLE of θ_{ij}, $1 \le i, j \le M$, given the observations, $(Y_1, \ldots, Y_n) = (y_1, \ldots, y_n)$, and given that the initial state is $Y_0 = y_0$.

Next, consider a Markov chain consisting of two states - state 1 and state 2. If the sequence of observed states is 1,1,1,2,1,2,2,1,1,2,2,2,1,2,1,2, use the results obtained above to obtain the MLE of $\theta_{ij}, 1 \le i, h \le 2$.

9. The $\mathbb{R}$-valued observation r.v. Y has (marginal) probability density function

$$f(y) = \begin{cases} \frac{1}{2}, & -1 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

The parameter θ to be estimated is a r.v. given by $\theta = Y^2$.

(i) Find $\hat{E}[\theta|Y]$, the LMSE of θ given Y.

(ii) Find $g_{MSE}(Y)$, the MMSE of θ given Y.

10. Let $Y^k \triangleq (Y_1, \ldots, Y_k)$ be a sequence of i.i.d. r.v.'s uniformly distributed on $[\theta_1, \theta_2]$, where θ_1 and θ_2 are unknown constants satisfying $0 \le \theta_1 < \theta_2$. Set $\theta \triangleq (\theta_1, \theta_2)$.

(i) Find $g^{\theta}_{ML,k}(Y^k) \triangleq \begin{bmatrix} g^{\theta_1}_{ML,k}(Y^k) \\ g^{\theta_2}_{ML,k}(Y^k) \end{bmatrix}$ – the MLE's of θ_1 and θ_2 given Y^k.

Next, consider $g^{\theta_1}_{ML,k}$ – the MLE of θ_1.

(ii) Determine whether or not $g^{\theta_1}_{ML,k}$ is unbiased, $k \ge 1$.

(iii) Determine if $\left\{g^{\theta_1}_{ML,k}\right\}_{k=1}^{\infty}$ is asymptotically unbiased.

(iv) Determine if $\left\{g^{\theta_1}_{ML,k}\right\}_{k=1}^{\infty}$ is consistent.

11. Let the observation be a $\mathbb{R}$-valued r.v. Y with (marginal) probability density function

$$f(y) = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0. \end{cases}$$

The parameter θ to be estimated is also a $\mathbb{R}$-valued r.v. whose conditional probability density function given the observation $Y = y$ is

$$G_y(t) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(t - y - \frac{y^2}{2})^2\right\}, -\infty < t < \infty.$$

(i) Find $\hat{E}[\theta|Y]$, the LMSE estimate of θ given Y.

(ii) Compute the error covariance $\mathrm{cov}\left[\theta - \hat{E}\left[\theta|Y\right]\right]$.

12. Let $Y_1, \ldots, Y_m$ be the $\mathbb{R}^+$-valued r.v.'s with (common)-probability density function

$$f_\theta(y) = \frac{1}{\theta}\exp\left(-\frac{y}{\theta}\right), \quad y > 0$$

where θ in $\mathbb{R}^+ (= (0, \infty))$ is an unknown constant.

(i) Find the MLE $g_{ML,n}(Y_1, \ldots, Y_n)$ of θ on the basis of $(Y_1, \ldots, Y_n)$.

(ii) Show whether or not $g_{ML,n}$ is a MVUE of θ.

13. Let the r.v. θ be uniformly distributed on $[0,1]$. Let Y be a $\{0,1\}$-valued r.v. with

$$P(Y=1|\theta=t) = t = 1 - P(Y=0|\theta=t), \quad 0 \leq t \leq 1.$$

(i) Find the MMSE estimate of θ given Y.

(ii) For each t in $[0,1]$, let $Y_1, \ldots, Y_n$ be i.i.d. r.v.'s with (common) distribution specified above. Determine the MAP estimate of θ given $(Y_1, \ldots, Y_n)$.

(iii) Compute the bias of the estimator in part (ii) above.

14. Let X_1 and X_2 be i.i.d. $\mathbb{R}$-valued r.v.'s. Let $\theta = X$, and $Y \triangleq X_1 + X_2$.

(i) Find the MMSE estimator of θ on the basis of Y.

(ii) If the r.v.'s X_1 and X_2 are Gaussian with zero mean and unit variance, compute the estimation error covariance $E[|\theta - g_{MSE}(Y)|^2]$.

15. Let the parameter θ be a $\mathbb{R}$-valued exponential rv with probability density function

$$g(t) = \alpha e^{-\alpha t}, \qquad t \geq 0,$$

where $\alpha > 0$ is given. The observation is modelled by a R-valued rv Y given by

$$Y = \theta X + N$$

where

- the rv's θ, X, N are mutually independent;
- $X \sim \mathcal{N}(1,1)$;
- $N \sim \mathcal{N}(0,1)$.

(i) Determine $\hat{E}[\theta|Y]$.

(ii) Determine the error covariance, $E[(\theta - \hat{E}[\theta|Y])^2]$, of the estimator in part (i).

16. This problem concerns the estimation of the *rate* of a Poisson process. Let $\{N(t), t \geq 0\}$, be a Poisson process with rate $\theta \geq 0$.

Consider first the non-Bayesian situation in which $\theta \geq 0$ is an *unknown constant*. The Poisson process is observed for a time-interval $[0,T]$, where $T > 0$ is *fixed*. Thus, the observations consist of $(M, Y_1, \ldots, Y_M)$, where M is a rv with values in $\{1, 2, \ldots\}$ denoting the number of events observed in $[0, T]$, and $Y_i \geq 0$ is a rv representing the interarrival time between the $(i-1)^{st}$ and i^{th} of these events, $i = 1, \ldots, M$.

(i) Determine a nontrivial sufficient statistic for the family of distributions $\{F(\theta), \theta \geq 0\}$, where $F(\theta)$ is the joint probability distribution of the rv's $(M, Y_1, \ldots, Y_M)$ as above corresponding to a Poisson process of rate θ.

(ii) Derive a maximum-likelihood estimator (MLE), $g_{ML,T}(M, Y_1, \ldots, Y_M)$, for θ on the basis of $(M, Y_1, \ldots, Y_M)$, for a *fixed value* of $T > 0$.

(iii) Determine a MVUE, $g_T(M, Y_1, \ldots, Y_M)$, for θ on the basis of $(M, Y, \ldots, Y_M)$, for a *fixed value* of $T > 0$.

(iv) Determine whether or not the estimators in (ii) and (iii) are consistent (in any sense) as $T \to \infty$.

Next, consider the Bayesian situation in which θ is an *exponential rv* with given parameter $\alpha > 0$, i.e.,

$$G(t) \triangleq P[\theta \leq t] = 1 - e^{-\alpha t}, \qquad t \geq 0.$$

(v) Let $T = 1$, i.e., the Poisson process is observed for 1 time unit. Determine the minimum mean-squared error estimator, $g_{MSE}(1, Y_1)$, of θ on the basis of $(1, Y_1)$.

[*Some useful facts*: Recall for a Poisson process $\{N(t), t \geq 0\}$ with rate $\theta \geq 0$ that:

- the interarrival times $\{Y_1, Y_2, \ldots\}$ are i.i.d. rv's, each exponential with parameter θ; so, the probability that no event occurs in a time-interval of length Δt is $e^{-\theta \Delta t}$;
- the rv $N(t), t \geq 0$, denoting the number of events occurring in the interval $[0, t]$, is a Poisson rv with mean θt.]

17. (i) Let Y_1, Y_2 be $\mathbb{R}$-valued i.i.d. rv's with (common) proabability density function

$$f_\theta(y) = \frac{1}{2} \exp[-|y - \theta|], \quad -\infty < \theta < \infty, \quad -\infty < y < \infty,$$

where θ is an *unknown constant.* Find an *unbiased* maximum-likelihood estimate $g_{ML}(Y_1, Y_2)$ of θ on the basis of (Y_1, Y_2).

(ii) Next, consider a Bayesian estimation problem where the parameter θ is a rv distributed uniformly on the interval $[-1, 1]$. The observation consists of a (single) rv Y whose conditional density function, given $\theta = t$, is

$$f_t(y) = \frac{1}{2} \exp[-|y - t|], \quad -1 \leq t \leq 1, \quad -\infty < y < \infty.$$

Determine the *maximum aposteriori estimate* $g_{MAP}(Y)$ of θ on the basis of Y.

18. (i) For each $\theta > 0$, let $Y_1, \ldots, Y_n$ be $\mathbb{R}$-valued i.i.d. rv's, each distributed uniformly on the *interval* $[-\theta, \theta]$. Find the maximum-likelihood estimate $g_{ML,n}(Y^n)$ of θ on the basis of $Y^n \triangleq (Y_1, \ldots, Y_n)$.

(ii) Is the sequence of estimates $\{g_{ML,n}(Y^n)\}_{n=1}^{\infty}$ asymptotically normal? Substantiate your answer.

(iii) Next, consider a Bayesian situation in which the parameter θ is a rv with values in the set $\{1, 2\}$ where

$$P[\theta = 1] = P[\theta = 2] = \frac{1}{2}.$$

The observation consists of a (single) rv Y, whose conditional distribution, given $\theta = t$, is uniform on the *interval* $[-t, t], t = 1, 2$. Given that $Y = 1$ is observed, determine the mean-error magnitude (MEM) estimate of θ on the basis of $Y = 1$.

19. A signal with finite energy, modelled by a rv θ with $E[\theta^2] < \infty$, is observed in additive noise as follows:

$$Y_t = \theta + N_t, \quad t = 1, 2, \ldots,$$

where $\{N_t\}_{t=1}^{\infty}$ is a white noise process with

$$E[N_t] = 0, \quad E[N_t N_s] = \sigma^2 \delta(t, s), \quad t, s = 1, 2, \ldots$$

where $\sigma^2 > 0$, and $\delta(t, s) = 1$ if $t = s$ and 0 if $t \neq s$. Assume further that θ and $\{N_t\}_{t=1}^{\infty}$ are *uncorrelated.*

For $t = 1, 2, \ldots$, consider the following two estimates of θ on the basis of $Y^t \triangleq (Y_1, \ldots, Y_t)$:

$$\hat{\theta}_t(Y^t) \triangleq \hat{E}[\theta | Y^t], \text{ the linear least} - \text{squares error estimate,}$$

and

$$\overline{\theta}_t(Y^t) \triangleq \frac{1}{t} \sum_{\ell=1}^{t} Y_\ell, \text{ the sample} - \text{mean estimate.}$$

(i) By using only the orthogonality principle, **disprove** the claim that

$$\hat{\theta}_t(Y^t) = \overline{\theta}_t(Y^t) \quad P - a.s., \qquad t = 1, 2, \ldots.$$

(ii) Is the sequence of estimators $\{\hat{\theta}_t\}_{t=1}^{\infty}$ *consistent* in any sense?