# ENEE 621: Estimation and Detection

**Solutions: Problem Set 4** **Spring 2007**

**Narayan**

[1]

(a) $\sum_{Y_1} = E[Y_1 - E[Y_1])^2] = E[\{(1+N_1)\theta - 1\}^2] = \ldots = 5$ (check)

Hence, $\hat{E}[\theta|Y_1] = \mu_\theta + \sum_{\theta Y_1} \sum_{Y_1}^{-1} (Y_1 - \mu_{Y_1})$.

$$\Sigma_{\theta Y_1} = E[(\theta - 1)(\theta + \theta N_1 - 1)] = 2.$$

$$\Rightarrow \hat{E}[\theta|Y_1] = 1 + \frac{2}{5}(Y_1 - 1) = \frac{3}{5} + \frac{2}{5}Y_1.$$

(b)

$$\begin{aligned} \hat{E}[\theta - 1|Y_1, Y_2] &= \hat{E}[\theta - 1|Y_1 Y_2 - \hat{E}[Y_2|Y_1]] \\ &= \hat{E}[\theta - 1|Y_1] + \hat{E}[\theta - 1|Y_2 - \hat{E}[Y_2|Y_1]] \end{aligned}$$

i.e.

$$\hat{E}[\theta|Y_1, Y_2] = 1 + \hat{E}[\theta|Y_1] - 1 + \hat{E}[\theta|Y_2 - \hat{E}[Y_2]Y_1]] - 1$$

i.e.,

$$\hat{E}[\theta|Y_1, Y_2] = \hat{E}[\theta|Y_1] - 1 + \hat{E}[\theta|Y_2 - \hat{[}Y_2|Y_1]]. \qquad (*)$$

We now compute $\hat{E}[\theta|Y_2 - \hat{E}[Y_2|Y_1]]$ in (*). First observe that

$$\begin{aligned} \hat{E}[Y_2|Y_1] &= \hat{E}[(1 + N_2)\theta|Y_1] = \hat{E}[\theta|Y_1] + \hat{E}[\theta N_2|Y_1] \\ &= \frac{3}{5} + \frac{2}{5}Y_1 + \hat{E}[\theta N_2|Y_1], \text{ from (a)} \end{aligned} \qquad (\$)$$

In (\$), note that since $\sum_{\theta N_2, Y_1} = E\left[\theta N_2(Y_1 - 1)\right]$

$$\begin{aligned} &= E[\theta N_2 Y_1] = E[\theta N_2 \theta(1 + N_1)] = E[\theta^2 N_2(1 + N_1)] \\ &= E[\theta^2]E[N_2 + N_1 N_2], \text{ because } \theta \text{ is independent of } (N_1, N_2) \\ &= E[\theta^2] \cdot [0 + 0] = 0, \text{ because } E[N_1 N_2] = E[N_1]E[N_2] = 0 \end{aligned}$$

we have $\hat{E}[\theta N_2|Y_1] = E[\theta N_2] = E[\theta]E[N_2] = 0$. Hence, in ($),

$$\hat{E}[Y_2|Y_1] = \frac{3}{5} + \frac{2}{5}Y_1.$$

$$\Rightarrow Y_2 - \hat{E}[Y_2|Y_1] = Y_2 - \frac{3}{5} - \frac{2}{5}Y_1 \triangleq \tilde{Y}.$$

Then in (*),

$$\begin{aligned}\hat{E}[\theta|Y_2 - \hat{E}[Y_2|Y_1]] &= \hat{E}[\theta|\tilde{Y}] \\ &= \mu_\theta + \Sigma_{\theta\tilde{Y}}\Sigma_{\tilde{Y}}^{-1}(\tilde{Y} - \mu_{\tilde{Y}}), \quad (\mu_{\tilde{Y}} = 0, \quad \textit{why?}) \\ &= \mu_\theta + \Sigma_{\theta\tilde{Y}}\Sigma_{\tilde{Y}}^{-1}\tilde{Y}\end{aligned}$$

Verify that

$$\Sigma_{\theta\tilde{Y}} = \frac{6}{5}, \Sigma_{\tilde{Y}} = \frac{79}{25}, \quad (?!)$$

$$\begin{aligned}\Rightarrow E[\theta|\tilde{Y}] &= 1 + \frac{6}{5}\cdot\frac{25}{79}(Y_2 - \frac{3}{5} - \frac{2}{5}Y_1) \\ &= 1 + \frac{30}{79}(Y_2 - \frac{3}{5} - \frac{2}{5}Y_1)\end{aligned}$$

Thus, in (*)

$$\hat{E}[\theta|Y_1, Y_2] = \hat{E}[\theta|Y_1] + \frac{30}{79}(Y_2 - \frac{3}{5} - \frac{2}{5}Y_1).$$

[2]

(a)(b) Note that if we had replaced $\theta_0 \perp \{W_t\}_0^\infty$ by the condition that $\theta_0$ is independent of $\{W_t\}_0^\infty$ in the problem statement, we'd have obtained that $(\theta_t, Y^t)$ is Gaussian as also $(\theta_{t+T}, Y^t)$. Then $E[\theta_t|Y^t] = \hat{E}[\theta_t|Y^t]$ and $E[\theta_{t+T}|Y^t] = \hat{E}[\theta_{t+T}|Y^t]$. Returning to the problem at hand, I'll provide the initial steps below – the rest can be obtained as in class albeit with more tedious calculations. Let $\hat{\theta}_{t|t} \triangleq \hat{E}[\theta_t|Y^t], \hat{\theta}_{t+T/t} \triangleq \hat{E}[\theta_{t+T}|Y^t], T \geq 1$. Observe that

$$\theta_{t+T} = \theta_{t+1} + b\sum_{i=t+1}^{t+T-1} W_i$$

$$\Rightarrow \hat{\theta}_{t+T|t} = \hat{\theta}_{t+1|t} + b \sum_{i=t+1}^{t+T-1} \hat{E}[W_i|Y^t].$$

Since $Y^t \epsilon$ span $\{\theta_0, \{W_s\}_0^t\}$, we have for $i \geq t+1$ that $W_i \perp Y^t$, so that $\hat{E}[W_i|Y^t] = E[W_i] = 0$.

$$\Rightarrow \hat{\theta}_{t+T|t} = \hat{\theta}_{t+1|t} \quad \forall t = 0, 1, \ldots \text{ and for all } T \geq 1.$$

So the problem now becomes one of finding recursions for $\hat{\theta}_{t+1|t}$ and $\hat{\theta}_{t|t}$ as in class.

As in class, it follows that $\nu_t = Y_t - E[Y_t|Y^{t-1}] = Y_t - \hat{\theta}_{t/t-1}$ with $E[\nu_t] = 0, E[\nu_t \nu_s] = \left[\sum_{t/t-1} + Q_t\right] \delta_{st}$.

**Assume that** $Q_t > 0 \quad \forall t$

Next: $\theta_{t+1} = \theta_t + bW_t$, so that

$$\hat{\theta}_{t+1|t} = \hat{\theta}_{t|t} + b\hat{E}[W_t|Y^t]. \qquad (*)$$

Now,

$$\begin{aligned}
\hat{E}[W_t|Y^t] &= \hat{E}[W_t|Y^{t-1}, \nu_t] = \hat{E}[W_t|Y^{t-1}] + \hat{E}[W_t|\nu_t], \text{ (notice } \hat{E}[W_t|Y^{t-1}] = 0 \text{ why?)} \\
&= \Sigma_{W_t,\nu_t} \left[\Sigma_{t|t-1} + Q_t\right]^{-1} \nu_t \\
&= E[W_t(Y_t - \hat{E}[Y_t|Y^{t-1}]) \left[\Sigma_{t|t-1} + Q_t\right]^{-1} \nu_t
\end{aligned}$$

Notice that

$$E[W_t(Y_t - \hat{E}[Y_t|Y^{t-1}]) = E[W_t Y_t] - E[W_t \hat{E}[Y_t|Y^{t-1}]]$$

$$\text{and } E[W_t \hat{E}[Y_t|Y^{t-1}]] = 0, \quad (why?)$$

Also,

$$\begin{aligned}
&E[W_t Y_t] \\
&= E[W_t(\theta_t + W_t)] \\
&= E[W_t^2] + E[W_t \theta_t], \quad (E[W_t \theta_t] = 0, why?) \\
&= Q_t
\end{aligned}$$

Therefore,

$$\hat{E}[W_t|Y^t] = Q_t\left[\Sigma_{t|t-1} + Q_t\right]^{-1}\nu_t$$

Thus, in (*)

$$\hat{\theta}_{t+1|t} = \hat{\theta}_{t|t} + bQ_t\left[\Sigma_{t|t-1} + Q_t\right]^{-1}\nu_t.$$

Now, proceed as in class. In particular, you'll get:

$$\hat{\theta}_{t+1|t} = \hat{\theta}_{t|t-1} + K_t\nu_t$$

where

$$K_t = \left(\Sigma_{t|t-1} + bQ_t\right)\left(\Sigma_{t|t-1} + Q_t\right)^{-1}.$$

There will be appropriate modifications in the recursions for $\sum_{t+1|t}$ (in terms of $\sum_{t|t}$) and $\sum_{t|t}$ (in terms of $\sum_{t|t-1}$).

[3] Not true. For instance, let $Y \perp Z$ with $\mu_Y = 0$. Then $\hat{E}[Y|Z] = \mu_Y = 0$. Then RHS $= \hat{E}[\theta|Y, 0] = \hat{E}[\theta|Y] \neq \hat{E}[\theta|Y, Z] =$ LHS. (Is $Y \perp Z$ the only case when LHS $\neq$ RHS?).

[4] $\mu_{X_n} = 0$.

$$\begin{aligned}
\Sigma_{X_n,(X_{n-1},X_{n+1})} &= E[X_n \cdot [X_{n-1} \quad X_{n+1}]] \\
&= [E[X_n X_{n-1}] \quad E[X_n X_{n+1}]] \\
&= [R_1 \quad R_1], \text{ since } R_i = R_{-i}, \quad i = 1, 2, \ldots. \\
\Sigma_{(X_{n-1},X_{n+1})} &= E\left[\begin{bmatrix} X_{n-1} \\ X_{n+1} \end{bmatrix} [X_{n-1} \quad X_{n+1}]\right] \\
&= \begin{bmatrix} R_0 & R_2 \\ R_2 & R_0 \end{bmatrix}.
\end{aligned}$$

Since $R_0 \neq R_2$ by hypothesis, we see that $\Sigma^{-1}_{(X_{n-1},X_{n+1})}$ exists, and

$$\Sigma^{-1}_{(X_{n-1},X_{n+1})} = \frac{1}{(R_0^2 - R_2^2)}\begin{bmatrix} R_0 & -R_2 \\ -R_2 & R_0 \end{bmatrix}$$

Hence,

$$\begin{aligned}
&\hat{E}[X_n|X_{n-1}, X_{n+1}] \\
&= \frac{1}{(R_0^2 - R_2^2)}[R_1 \quad R_1] \begin{bmatrix} R_0 & -R_2 \\ -R_2 & R_0 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ X_{n+1} \end{bmatrix} \\
&= \left( \frac{R_1}{R_0 + R_2} \right) (X_{n-1} + X_{n+1}).
\end{aligned}$$

[5] (i)

$$f_t(y^n) = t^{-n} u(t - \max_{i \le i \le n} y_i) \cdot u(\min_{1 \le i \le n} y_i). \tag{$*$}$$

$$\nu(t) = \frac{1}{b-a}, \quad a \le t \le b. \tag{$**$}$$

Hence,

$$\nu_{y^n}(t) = \frac{f_t(y^n)\nu(t)}{f(y^n)}, \quad a \le t \le b, \tag{$*^3$}$$

where

$$\begin{aligned}
f(y^n) &= \int_a^b f_t(y^n)\nu(t)dt \\
&= \frac{1}{(b-a)} \int_a^b t^{-n} \cdot u(t - \max_i y_i) \cdot u(\min_i y_i) dt \\
&= \frac{u(\min_i y_i)}{(b-a)} \int_{\max\{a, \max_i y_i\}}^b u(b - \max\{a, \max_i y_i\}) dt
\end{aligned}$$

i.e.,

$$\begin{aligned}
f(y^n) &= \frac{u(\min_i y_i) u(b - \max\{a, \max_i y_i\})}{(b-a)} \int_{\max\{a, \max_i y_i\}}^b t^{-n} dt \\
&= \frac{u(\min_i y_i) u(b - \max\{a, \max_i y_i\})}{(b-a)(n-1)} \qquad (*^4) \\
&\times \left[ \left( \max\{a, \max_i y_i\} \right)^{-(n-1)} - b^{-(n-1)} \right].
\end{aligned}$$

The desired form of $\nu_{y^n}(t)$ follows upon substituting (*), (**), and ($*^4$) in ($*^3$).

(ii) From (i), the posteriori density function $\nu_{y^n}(t), a \le t \le b$, has a unique maximum at

$$t = \max\{a, \max_i y_i\},$$

so that

$$g_{MAP}(y^n) = \max\{a, \max_{i \le i \le n} y_i\}.$$

(iii) Consider first the conditional expectation

$$\begin{aligned}
E\left[g_{MAP}(Y^n)|\theta\right] &= E\left[\max\{a, \max_i Y_i\}|\theta\right] \\
&= aP_\theta\left(\max_i Y_i < a\right) + \int_{u \ge a} u \left(\frac{d}{du} P_\theta\left(\max_i Y_i \le u\right)\right) du \\
&= \frac{a \cdot a^n}{\theta^n} + \int_a^\theta u \frac{d}{du}\left(\frac{u^n}{\theta^n}\right) du = \frac{a^{n+1}}{\theta^n} + \frac{n}{\theta^n}\int_a^\theta u^n du \\
&= \frac{a^{n+1}}{\theta^n} + \frac{n}{\theta^n}\left(\frac{\theta^{n+1} - a^{n+1}}{n+1}\right) = \frac{a^{n+1}}{n+1}\theta^{-n} + \frac{n}{n+1}\theta \\
\Rightarrow \ \text{Bias} &= E\left[g_{MAP}(Y^n) - \theta\right] = E\left[E\left[g_{MAP}(Y^n)|\theta\right] - \theta\right] \\
&= E\left[\frac{a^{n+1}}{n+1} \cdot \theta^{-n} + \frac{n}{n+1}\theta - \theta\right] \\
&= \frac{1}{(n+1)} E\left[a^{n+1}\theta^{-n} - \theta\right] = \frac{1}{(n+1)}\left[a^{n+1}\int_a^b \frac{\theta^{-n}}{(b-a)} d\theta - \frac{(a+b)}{2}\right] \\
&\ \vdots \\
&= \frac{a}{(n^2-1)(b-a)}\left[a - b\left(\frac{a}{b}\right)^n\right] - \frac{a+b}{2(n+1)}.
\end{aligned}$$

Note that $g_{MAP}$ is asymptotically unbiased.

(iv)

$$g_{MSE}(y^n) = E[\theta|Y^n = y^n] = \int_a^b t\nu_{y^n}(t)dt$$

$$= \frac{n-1}{\left[(\max\{a, \max_i y_i\})^{(n-1)} - b^{-(n-1)}\right] u\left(b - \max\{a, max_i y_i\}\right)}, \text{ from part } (i)$$

$$\times \int_{\max\{a, \max_i y_i\}}^b t \cdot t^{-n} dt \quad u\left(b - \max\{a, max_i y_i\}\right)$$

$$= \left(\frac{n-1}{n-2}\right) \frac{(\max\{a, \max_i y_i\})^{-(n-2)} - b^{-(n-2)}}{(\max\{a, \max_i y_i\})^{-(n-1)} - b^{-(n-1)}}$$

(v)

$$E[\theta] = \frac{a+b}{2} = \frac{3}{2}; \quad E[Y_1|\theta] = \frac{\theta}{2} \Rightarrow E[Y_1] = \frac{3}{4}.$$

$$\Sigma_{\theta Y_1} = E[\theta Y_1] - E[\theta]E[Y_1]$$

$$= E\left[E[\theta Y_1|\theta]\right] - \frac{9}{8}, \text{ notice that } E[\theta Y_1|\theta] = \theta^2/2$$

$$= E[\frac{\theta^2}{2}] - \frac{9}{8} = \frac{1}{6}\left[b^2 + ab + a^2\right] - \frac{9}{8} = \frac{7}{6} - \frac{9}{8} = \frac{1}{24}.$$

$$\Sigma_{Y_1} = E[Y_1^2] - \frac{9}{16} = E[E[Y_1^2|\theta]] - \frac{9}{16} = E\left[\frac{\theta^3}{3}\right] - \frac{9}{16}$$

$$= (b^2 + ab + a^2)/9 - 9/16 = 31/144$$

$$\Rightarrow \hat{E}[\theta|Y_1] = \frac{3}{2} + \frac{1}{24} \cdot \frac{144}{31}(Y_1 - \frac{3}{4})$$

$$= \frac{3}{2} + \frac{6}{31}\left(Y_1 - \frac{3}{4}\right).$$