ENEE 621: ESTIMATION AND DETECTION

Spring 2007

Problem Set 4 **Narayan**

1. Let θ be a $\mathbb{R}$-valued Gaussian r.v. with mean 1 and variance 2. Two observations Y_1 and Y_2 of θ in noise are made as follows:

$$Y_1 = (1 + N_1)\theta, \qquad Y_2 = (1 + N_2)\theta,$$

where θ is independent of (N_1, N_2), $E[N_1] = E[N_2] = 0, E[N_1^2] = E[N_2^2] = 1$, and $N_1 \perp N_2$.

(i) Find $\hat{E}[\theta|Y_1]$.

(ii) Find $\hat{E}[\theta|Y_1, Y_2]$ in terms of $\hat{E}[\theta|Y_1]$ and Y_2 in a recursive fashion.

2. Consider the one-dimensional system

$$\theta_{t+1} = \theta_t + bW_t, \qquad t = 0, 1, \ldots$$

where the observation process is

$$Y_t = \theta_t + W_t, \quad t = 0, 1, \ldots.$$

Here, $\{W_t\}_0^\infty$ is a sequence of $\mathbb{R}$-valued Gaussian r.v.'s with $E[W_t] = 0, E[W_t W_s] = Q_t \delta_{ts}$, where $Q_t > 0, t, s = 0, 1, \ldots$. Let $\theta_0 \sim \mathcal{N}(0, \sum_0)$, and assume that $\theta_0 \perp \{W_t\}_0^\infty$; $b \neq 0$ is a known constant. Let $Y^t \triangleq (Y_0, \ldots, Y_t)$.

(a) Find a recursive scheme for generating $\hat{E}[\theta_t|Y^t], t = 0, 1, \ldots$.

(b) Find a recursive scheme for generating $\hat{E}[\theta_{t+T}|Y^t]$, where T is a positive integer.

3. Let θ, Y and Z be $\mathbb{R}$-valued r.v.'s with $E[|\theta|^2] < \infty$, $E[|Y|^2] < \infty$, $E[|Z|^2] < \infty$. It is true that

$$\hat{E}[\theta|Y, Z] = \hat{E}[\theta|Y, \hat{E}[Y|Z]].$$

(If true, provide a formal proof; if not, give a counter-example.)

4. Let $\{X_n\}_{n=1}^\infty$ be a sequence of $\mathbb{R}$-valued, zero-mean r.v.'s with $E[X_n X_m] = R_{n-m}$ for all $n, m \geq 1$. Assuming that $R_{n-m} > 0$ for all $n, m \geq 1$, and that $R_0 \neq R_2$, find $\hat{E}[X_n|X_{n-1}, X_{n+1}]$ – the best **linear interpolator**.

5. Let $Y^n \triangleq (Y_1, \ldots, Y_n)$ be a sequence of i.i.d. $\mathbb{R}$-valued r.v.'s, each distributed uniformly on $[0, \theta]$, where θ is itself a $\mathbb{R}$-valued r.v. distributed uniformly on $[a, b]$, where a, b are known constants with $0 < a < b$.

Let $u(\cdot)$ denote the unit step function on $\mathbb{R}$, i.e.,

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) Show that the conditional probability density function of θ given $Y^n = y^n$, where $y^n = (y_1, \ldots, y_n)$, is

$$\gamma_{y^n}(t) = \frac{(n-1)t^{-n} u(t - \max_{1 \leq i \leq n} y_i)}{\left[\{\max(a, \max_{1 \leq i \leq n} y_i)\}^{-(n-1)} - b^{-(n-1)}\right] u\left(b - \max\left(a, \max_{1 \leq i \leq n} y_i\right)\right)}$$

(ii) Find the MAP estimate of θ given $Y^n = y^n$.

(iii) For the estimator of part (ii) above, compute the bias $E[g_{MAP}(Y^n) - \theta]$.

(iv) Show that the minimum mean-squared error estimate of θ given $Y^n = y^n$ is

$$g_{MSE}(y^n) = \left(\frac{n-1}{n-2}\right) \frac{\{\max(a, \max_{1 \leq i \leq n} y_i)\}^{-(n-2)} - b^{-(n-2)}}{\{\max(a, \max_{1 \leq i \leq n} y_i)\}^{-(n-1)} - b^{-(n-1)}}.$$

(v) Let $a = 1, b = 2$. Find the linear least-squares error estimate $\hat{E}[\theta | Y_1]$ of θ given the single observation Y_1.