ENEE 621: ESTIMATION AND DETECTION THEORY

Problem Set 5: Solutions **Spring 2007**

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1. Note that Y_1 is independent of $\{X_n\}_{n=1}^{\infty}$. Now, observe that H_0 is a composite hypothesis, and we are in a Bayesian situation with the rv θ (= the rv P_0) taking values in $\Theta_0 = \{1/4, 3/4\}$ with $P_0[\theta = \frac{1}{4}] = \frac{1}{4} = 1 - P_0[\theta = \frac{3}{4}]$, where P_0 denotes the conditional pmf of θ given $H = H_0$. H_1 is a simple hypothesis.

   Under H_0: For each $t \in \Theta_0 = \{\frac{1}{4}, \frac{3}{4}\}$, $\{Y_n\}_{n=1}^{\infty}$ is a 1st order Markov process with transition probabilities:

$$\begin{aligned} P_t\left[Y_{n+1} = 0 | Y_n = 0\right] &= P_t\left[X_n = 0 | Y_n = 0\right] \\ &= P_t[X_n = 0] \quad (\textit{why?}) \\ &= t \end{aligned}$$

   Similarly,

$$P_t\left[Y_{n+1} = 0 | Y_n = 0\right] = P_t\left[Y_{n+1} = 0 | Y_n = 1\right] = 1 - t,$$

   and

$$P_t\left[Y_{n+1} = 1 | Y_n = 1\right] = t.$$

   Let $Y = (Y_1, \ldots, Y_n)$. Given a sequence of observations $y = (y_1, \ldots, y_N)$ where $P_0[Y_1 = 0] = 1$, we have for each $t \in \{1/4, 3/4\}$: $f_t(y) = P_t[Y = y] = (1-t)^{\tilde{N}} t^{N-1-\tilde{N}}$, where $\tilde{N}$ is a rv with values in $\{0, 1, \ldots, N-1\}$ denoting the number of transitions from "0" to "1" and from "1" to "0" in $y = (y_1, \ldots, y_N)$ (with $y_1 = 0$). Then:

$$\begin{aligned} \tilde{f}_0(y) &= f_{\frac{1}{4}}(y) P_0[\theta = 1/4] + f_{\frac{3}{4}}(y) P_0[\theta = 3/4] \\ &= \left(\frac{3}{4}\right)^{\tilde{N}} \left(\frac{1}{4}\right)^{N-1-\tilde{N}} \left(\frac{1}{4}\right) + \left(\frac{1}{4}\right)^{\tilde{N}} \left(\frac{3}{4}\right)^{N-1-\tilde{N}} \left(\frac{3}{4}\right) \\ &= \left(\frac{1}{4}\right)^{N} \left[3^{\tilde{N}} + 3^{N-\tilde{N}}\right]. \end{aligned}$$

   Under H_1: $\{Y_n\}_{n=1}^{\infty}$ is a 1st order Markov process with transition probabilities

$$P_1[Y_{n+1} = 0 | Y_n = 0] = P_1[Y_{n+1} = 0 | Y_n = 1] = 1/2$$

Then, for $P_1[Y_1 = 0] = 1$, we have

$$f_1(y) = P_1[Y = y] = \left(\frac{1}{2}\right)^{\tilde{N}} \left(\frac{1}{2}\right)^{N-1-\tilde{N}} = \left(\frac{1}{2}\right)^{N-1}.$$

Then:

$$(\ell rt)_\eta : \tilde{L}(y) = \frac{f_1(y)}{\tilde{f}_0(y)} \underset{H_1}{\overset{H_0}{\lessgtr}} \eta$$

i.e.,

$$\frac{\left(\frac{1}{2}\right)^{N-1}}{\left(\frac{1}{4}\right)^N \left[3^{\tilde{N}} + 3^{N-\tilde{N}}\right]} \underset{H_1}{\overset{H_0}{\lessgtr}} \eta$$

i.e.,

$$3^{\tilde{N}} + 3^{N-\tilde{N}} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{2^{N+1}}{\eta}$$

The LHS is the test statistic, and the RHS is the threshold.

(b) $g_{MAP}(y) = arg\max_{\theta\in\{1/4,3/4\}} g_y(\theta)$. Since $g_y(\theta) = \frac{f_\theta(y)g(\theta)}{f(y)}$, it suffices to maximize the numerator as a function of θ.

For $\theta = \frac{1}{4}$, $f_\theta(y)g(\theta) = (1-\frac{1}{4})^{\tilde{N}} \left(\frac{1}{4}\right)^{N-1-\tilde{N}} \cdot \frac{1}{4} = 3^{\tilde{N}}4^{-N}$.

For $\theta = \frac{3}{4}$, $f_\theta(y)g(\theta) = \left(1-\frac{3}{4}\right)^{\tilde{N}} \left(\frac{3}{4}\right)^{N-1-\tilde{N}} \cdot \frac{3}{4} = 3^{N-\tilde{N}}4^{-N}$.

$$\Rightarrow g_{MAP}(y) = \begin{cases} \frac{1}{4}, & \text{if } \tilde{N} \geq N - \tilde{N} \;\; \text{i.e. if } \tilde{N} \geq N/2 \\ \frac{3}{4}, & \text{if } \tilde{N} \leq \frac{N}{2}. \end{cases}$$

2. Given $c(0,0) = c(1,1) = 0$, and $\nu_0 = c(0,1) = 2c(1,0) = 2\nu_1$. For the Bayes' decision rule $d^* : \mathcal{Y}^* = \{y \in \mathbb{R} : d^*(y) = 0\} = \{y \in \mathbb{R} : h(y) < 0\}$, where $h(y) = \nu_1 p f_1(y) - 2\nu_1(1-p)f_0(y)$. Clearly for $|y| \geq 1$, $h(y) = \nu_1 p f_1(y) \geq 0 \Rightarrow d^*(y) = 1$ for $|y| \geq 1$. Next observe that $f_1(y) = 0 \Rightarrow f_0(y) = 0$, so that

$$\ell\text{rt} : \frac{f_0(y)}{f_1(y)} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\nu_1 p}{2\nu_1(1-p)} = \frac{p}{1-p}.$$

We are only concerned with $|y| < 1$ now, in which case

$$\ell rt : \frac{1 - |y|}{2 - |y|} \underset{H_1}{\overset{H_0}{\gtrless}} \frac{1}{8} \cdot \frac{p}{1-p},$$

which simplifies to $|y|(9p-8) \underset{H_1}{\overset{H_0}{\gtrless}} (10p-8)$. so that for $|y| < 1$:

$$|y| \underset{H_1}{\overset{H_0}{\gtrless}} \frac{10p-8}{9p-8}, \text{ if } p \in \left(\frac{8}{9}, 1\right)$$

$$|y| \underset{H_0}{\overset{H_1}{\gtrless}} \frac{10p-8}{9p-8}, \text{ if } p \in \left(0, \frac{8}{10}\right)$$

and always say H_1 if $p \in \left[\frac{8}{10}, \frac{8}{9}\right]$.

3. H_0 is a simple hypothesis whereas H_1 is composite. **Fix** $p_1 \neq p_0$ and consider the simple hypothesis testing problem that results. Consider the Neyman-Pearson test of size α. In a manner similar to Prob. 1, we get:

$$\alpha^{NP(\alpha;p_1)} : \frac{(1-p_1)^{\tilde{N}} p_1^{N-1-\tilde{N}}}{(1-p_0)^{\tilde{N}} p_0^{N-1-\tilde{N}}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta(\alpha; p_1)$$

where $\tilde{N} \in \{0, \ldots, N-1\}$ is a r.v. representing the # of transitions from "0" to "1" and from "1" to "0", and $\eta(\alpha; p_1)$ is the corresponding threshold. Upon simplification

$$d^{NP(\alpha;p_1)} : \tilde{N} \log \frac{(1-p_1)p_0}{(1-p_0)p_1} \underset{H_0}{\overset{H_1}{\gtrless}} \log \eta(\alpha; p_1) + (N-1) \log \left(\frac{p_0}{p_1}\right)$$

If $p_0 > p_1$: Then by $\frac{(1-p_1)p_0}{(1-p_0)p_1} > 0$.

$$d^{NP(\alpha;p_1)} : \tilde{N} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\log \eta(\alpha; p_1) + (N-1) \log \left(\frac{p_0}{p_1}\right)}{\log(1-p_1)p_0 - \log(1-p_0)p_1}$$

with $\eta(\alpha; p_1)$ such that

$$P\left(\tilde{N} \geq \frac{\log \eta(\alpha; p_1) + (N-1) \log \left(\frac{p_0}{p_1}\right)}{\log(1-p_1)p_0 - \log(1-p_0)p_1} \mid H = 0\right) = \alpha$$

$$\text{and let } \nu = \frac{\log \eta(\alpha; p_1) + (N-1)\log\left(\frac{p_0}{p_1}\right)}{\log(1-p_1)p_0 - \log(1-p_0)p_1}$$

(assuming α is such that solution $\exists$).

Under H_0, the statistics of $\tilde{N}$ depend on p_0 so that $\nu = \nu(\alpha; p_0)$, i.e., ν does not depend on p_1. Then

$$\mathcal{Y}_d^{NP(\alpha;p_1)} = \{y \in \{0,1\}^N : \tilde{N}(y) < \nu(\alpha; p_0)\}.$$

If $p_0 < p_1$: Can show since $\log \frac{(1-p)p_0}{(1-p_0)p_1} < 0$ that $d^{NP(\alpha;p_1)} : \tilde{N} \underset{H_1}{\overset{H_0}{\gtrless}} \nu'(\alpha; p_0)$, where ν' does not depend on p_1

$$\Rightarrow \mathcal{Y}_d^{NP(\alpha;p_1)} = \{y \in \{0,1\}^N : \tilde{N}(y) > \nu'(\alpha; p_0)\}.$$

If $p_0 = p_1$:

$$\mathcal{Y}_d^{NP(\alpha;p_1)} = \begin{cases} \{y \in \{0,1\}^N : \tilde{N}(y) < \nu(\alpha, p_0)\} & \text{if } p_0 > p_1 \\ \{y \in \{0,1\}^N : \tilde{N}(y) > \nu'(\alpha, p_0)\} & \text{if } p_0 < p_1. \end{cases}$$

Clearly, if $\Theta_0 = \{p_0\}, \Theta_1 = \{p_1 \in (0,1) = p_1 > p_0\}$, a UMP test of size α exists.
If $\Theta_0 = \{p_0\}, \Theta_1 = \{p_1 \in (0,1); p_1 < p_0\}, \exists$ UMP test of size α. If $\Theta_0 = \{p_0\}, \Theta_1 = (0, p_0) \bigcup (p_0, 1)$, clearly we must know if $p_1 > p_0$ or $p_1 < p_0$ to execute $d^{NP(\alpha;p_1)}$; hence, no UMP exists. When the UMP test does exist, the test statistic $= \tilde{N}$.

4. Fix $\sigma_1^2 \neq \sigma_0^2$. Consider the corresponding simple hypothesis testing problem with the Neyman-Pearson test of size α. Recalling that $f_h(y) = \frac{y}{\sigma_h^2} e^{-y^2/\sigma_h^2}, y \geq 0, h = 0, 1$.

$$d^{NP(\alpha;\sigma_1^2)} = \frac{\sigma_0^2}{\sigma_1^2} e^{-y^2\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta(\alpha; \sigma_1^2),$$

where the threshold η depends on the size α and on σ_1^2. Upon simplification,

$$d^{NP(\alpha;\sigma_1^2)} = y^2(\sigma_1^2 - \sigma_0^2) \underset{H_0}{\overset{H_1}{\gtrless}} \left[\log \eta(\alpha; \sigma_1^2) + \log\left(\frac{\sigma_1^2}{\sigma_0^2}\right)\right] \sigma_0^2 \sigma_1^2.$$

As in problem 3:

(*) If $\sigma_1^2 > \sigma_0^2$: $\mathcal{Y}_{d^{NP(\alpha;\sigma_1^2)}} = \{y \in [0,\infty), y^2 < \nu(\alpha,\sigma_0^2)\}$ where $\nu(\alpha,\sigma_0^2)$ is the soln of $P(Y^2 \geq \nu | H=0) = \alpha$, and does not depend on σ_1^2.

(**) If $\sigma_0^2 > \sigma_1^2$: $\mathcal{Y}_{d^{NP(\alpha;\sigma_1^2)}} = \{y \in [0,\infty) : y^2 > \nu'(\alpha,\sigma_0^2)\}$ where $\nu'(\alpha,\sigma_0^2)$ (not depending on σ_1^2) solves $P(Y^2 \leq \nu' | H=0) = \alpha$.

(***) If $\sigma_0^2 \neq \sigma_1^2$: $\mathcal{Y}_{d^{NP(\alpha;\sigma_1^2)}} = \begin{cases} \{y \in [0,\infty) : y^2 < \nu(\alpha,\sigma_0^2)\} & \text{if } \sigma_1^2 > \sigma_0^2 \\ \{y \in [0,\infty) : y^2 > \nu'(\alpha,\sigma_0^2)\} & \text{if } \sigma_0^2 > \sigma_1^2. \end{cases}$

(a) $\Theta_0 = \{\sigma_0^2\}, \Theta_1 = (\sigma_0^2, \infty) \Rightarrow \exists$ UMP by (*)

(b) $\Theta_0 = \{\sigma_0^2\}, \Theta_1 = (0,\sigma_0^2) \bigcup (\sigma_0^2, \infty) \Rightarrow$ no UMP by (***)

(c) $\Theta_0 = \{\sigma_0^2\}, \Theta_1 = (0,\sigma_0^2) \Rightarrow \exists$ UMP by (**).

5. Let $(Y_1, \ldots, Y_N)$ represent N independent coin tosses with $Y_i = 1$ if head, 0 if tail, $1 \leq i \leq N$. The LRT is: $\ell rt_\eta : \frac{f_1(y_1,\ldots,y_N)}{f_0(y_1,\ldots,y_N)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$

$$\Rightarrow \frac{p^{\tilde{N}}(1-p)^{N-\tilde{N}}}{\left(\frac{1}{2}\right)^N} \underset{H_0}{\overset{H_1}{\gtrless}} \eta \ ,$$

where $\tilde{N} = \tilde{N}(Y)$ is a r.v. with values in $\{0, \ldots, N\}$ and represents the number of heads (note: $\tilde{N}(Y) = \sum_{i=1}^N Y_i$). Simplifying and using the fact that $p \in \left(\frac{1}{2}, 1\right)$, we have

$$\ell rt_\eta : \tilde{N} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\log \eta - N \log(2(1-p))}{\log\left(\frac{p}{1-p}\right)}.$$

Clearly $S_N = \tilde{N}$. Under each hypothesis, $\tilde{N}$ is binomial so that

$$P(\tilde{N} = k | H = 0) = \binom{N}{k} \left(\frac{1}{2}\right)^N, k = 0, \ldots, N,$$

$$P\left(\tilde{N} = k | H = 1\right) = \binom{N}{k} p^k (1-p)^{N-k}, k = 0, \ldots, N.$$

$$p_F\left(d^{NP(\alpha)}\right) = \alpha \Leftrightarrow P(N \geq \nu | H = 0) = \alpha,$$

where ν solves

$$\sum_{k=\lceil \nu \rceil}^{N} P(\tilde{N} = k | H = 0) = \alpha$$

assuming α is such that soln. exists

$$\Rightarrow \left(\frac{1}{2}\right)^N \sum_{k=\lceil \nu \rceil}^{N} \binom{N}{k} = \alpha \quad \Rightarrow \lceil \nu \rceil = \lceil \nu \rceil(\alpha), \quad \text{a fn of } \alpha,$$

$$\Rightarrow d^{NP(\alpha)} : \tilde{N} \underset{H_0}{\overset{H_1}{\gtrless}} \lceil \nu \rceil.$$

6.

$$\begin{aligned} H_0 &: Y_t = N_t \\ H_1 &: Y_t = s_t + N_t \end{aligned} \qquad t = 1, \ldots, K.$$

The $K \times K$-covariance matrix R_K for the noise process $\{N_t, t = 1, \ldots, k\}$ has entries $R_K(t, \tau) = t \wedge \tau, 1 \leq t, \tau \leq K$. Observe that

$$R_K = \begin{bmatrix} 1 & 1 & \ldots & & & 1 \\ 1 & 2 & 2 & \ldots & & 2 \\ 1 & 2 & 3 & \ldots & & 3 \\ \vdots & \vdots & & \vdots & \vdots & \\ 1 & 2 & 3 & 4 & \ldots & K \end{bmatrix}$$

It can be shown by induction that $det R_K = 1$ (clearly $det R_1 = det R_2 = 1$).

(a) From class notes:

$$d^{NP(\alpha)} : y^T R_K^{-1} s \underset{H_0}{\overset{H_1}{\gtrless}} \nu(\alpha)$$

where $y^T = (y_1, \ldots, y_k)$, $s = \begin{bmatrix} 1 \\ 1 \\ \ldots \\ 1 \end{bmatrix}$ with K elements. Now, $R_K^{-1} R_K = I_{K \times K}$; $s =$

$\begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}$ is the 1st column of R_K so that $R_K^{-1}s =$ 1st column of $I_{K\times K}$.

$$\Rightarrow y^T R_K^{-1} s = y_1 \quad \Rightarrow \quad d^{NP(\alpha)} : y_1 \underset{H_0}{\overset{H_1}{\gtrless}} \nu(\alpha)$$

where $\nu = \nu(\alpha)$ solves: $P(Y_1 > \nu | H = 0) = \alpha$ i.e., $1 - \Phi(\nu) = \alpha$, or $\Phi(\nu) = 1 - \alpha \Rightarrow \nu = x_{1-\alpha}$ (a function only of α; see example of composite hypothesis testing in class notes for notation : $x_\alpha = \Phi^{-1}(\alpha)$). Then,

$$\begin{aligned} p_D\left(d^{NP(\alpha)}\right) &= P(Y_1 \geq \nu | H = 1) \\ &= P(Y_1 - 1 \geq \nu - 1 | H = 1), \ \textit{where } Y_1 - 1 \sim \mathcal{N}(0,1) \\ &= 1 - \Phi(\nu - 1) = 1 - \Phi(x_{1-\alpha} - 1) \end{aligned}$$

(b) $d^{NP(\alpha)} : y^T R_K^{-1} s \underset{H_0}{\overset{H_1}{\gtrless}} \nu(\alpha)$, where $s = \begin{bmatrix} 1 \\ 2 \\ \cdots \\ k \end{bmatrix}$ is also the last column of R_K so that $R_K^{-1}s =$ last column of $I_{K\times K} \Rightarrow y^T R_K^{-1} s = y_K$. Then proceed in a manner similar to part (a). (Note: $E[N_K^2] = K$.)

7. (a) Let a "head" be the event 1, and a "tail" the event 0. Fix θ in $(0,1), \theta \neq 1/2$, and consider the following simply hypothesis testing problem. (The original problem is one of composite hypothesis testing which we shall get to shortly.)

$$H_0 : \{X_i\}_1^N i.i.d., P(X_i = 1 | H = 0) = \frac{1}{2}; \{Y_i\}_1^N i.i.d., P(Y_i = 1 | H = 0) = \theta$$

$$H_1 : \{X_i\}_1^N i.i.d., P(X_i = 1 | H = 0) = \theta; \{Y_i\}_1^N i.i.d., P(Y_i = 1 | H = 0) = 1/2.$$

(Thus, H_0 says the X-coin is fair, the Y-coin is biased with bias = fixed prob. $\theta \neq 1/2$; H_1 says that the Y-coin is fair, the X-coin is bias "θ".) Let $N_X = a$ r.v. in $\{0; \ldots, N\}$ denoting the # of heads of the X-coin, and $N_Y =$ a r.v. in $\{0, \ldots, N\}$ denoting the number of heads of the Y-coin. Observe that since $\{X_i\}_1^N$ is independent of $\{Y_i\}_1^N$, we have N_X independent of N_Y.

$$\ell rt_\eta : L(x,y) \underset{H_0}{\overset{H_1}{\gtrless}} \eta, \qquad \begin{array}{l} x = (x_1, \dots, x_N) \\ y = (y_1, \dots, y_N). \end{array}$$

where

$$L(x,y) = \frac{\theta^{N_x}(1-\theta)^{N-N_x}\left(\frac{1}{2}\right)^N}{\left(\frac{1}{2}\right)^N \theta^{N_Y}(1-\theta)^{N-N_Y}}$$

$$\Rightarrow \ell rt\eta : \frac{\theta^{N_X}(1-\theta)^{N-N_x}}{\theta^{N_Y}(1-\theta)^{N-N_Y}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta.$$

Now consider the composite hypothesis testing problem with a view to setting up the generalized ℓrt. Observe that

$$\Theta_0 = \{1/2\} \times \{1/2\}^c \text{ where } \{1/2\}^c \triangleq (0, 1/2) \bigcup (1/2, 1)$$

(so that $\theta_0 = (1/2, \theta)$ a pair of parameters, where $\theta \in, \{\frac{1}{2}\}^c$). Likewise $\Theta_1 = \{1/2\}^c \times \{1/2\}$, so that $\theta_1 = (\theta, 1/2)$. Then the generalized LRT is:

$$g\ell rt_\eta : \hat{L}(x,y) = \frac{\max_{\theta \in \{1/2\}^c} \theta^{N_X}(1-\theta)^{N_X}}{\max_{\theta \in \{1/2\}^c} \theta^{N_Y}(1-\theta)^{N_Y}}.$$

It is easily verified that the maximizing values of θ are:

$$\hat{\theta}(N_X) = N_X/N \text{ in the numerator}$$

$$\hat{\theta}(N_Y) = N_Y/N \text{ in the denominator}$$

both should differ from 1/2 for $g\ell rt_\eta$ to exist.

$$\Rightarrow g\ell rt_\eta : \hat{L}(x,y) = \frac{\left(\frac{N_X}{N}\right)^{N_X}\left(\frac{N-N_X}{N}\right)^{N-N_X}}{\left(\frac{N_Y}{N}\right)^{N_Y}\left(\frac{N-N_Y}{N}\right)^{N-N_Y}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

which simplifies to:

$$\frac{N_X^{N_X}(N-N_X)^{N-N_X}}{N_Y^{N_Y}(N-N_Y)^{N-N^Y}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

Next, to check for conditions for a UMP to exist: for a fixed θ in $\{1/2\}^c$, the Neyman-Pearson test of size α is:

$$d^{NP(\alpha;\theta)}: \ \theta^{N_X - N_Y}(1-\theta)^{N_Y - N_X} \underset{H_0}{\overset{H_1}{\gtrless}} \eta(\alpha, \theta)$$

i.e.,

$$\left(\frac{\theta}{1-\theta}\right)^{N_X - N_Y} \underset{H_0}{\overset{H_1}{\gtrless}} \eta(\alpha, \theta),$$

i.e.,

$$(N_X - N_Y)\log\left(\frac{\theta}{1-\theta}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \log\eta(\alpha, \theta).$$

Then as in probs. 3, 4;

(\$) If $\theta \in (\frac{1}{2}, 1) : d^{NP(\alpha;\theta)} : N_X - N_Y \underset{H_0}{\overset{H_1}{\gtrless}} \nu(\alpha)$,where ν solves:

$$P(N_X - N_Y \geq \nu | H = 0) = \alpha \text{ (assuming soln. } \exists).$$

(*) If $\theta \in (0, \frac{1}{2}) : d^{NP(\alpha,\theta)} : N_Y - N_X \underset{H_0}{\overset{H_1}{\gtrless}} \nu'(\alpha)$, where ν' solves

$$P(N_Y - N_X \geq \nu' | H = 0) = \alpha \text{ (assuming soln } \exists).$$

Now, if $\Theta_0 = \{\frac{1}{2}\} \times (\frac{1}{2}, 1), \Theta_1 = (\frac{1}{2}, 1) \times \{\frac{1}{2}\}$, UMP test exists as (\$) obtains. If $\Theta_0 = \{\frac{1}{2}\} \times (0, \frac{1}{2}), \Theta_1 = (0, \frac{1}{2}) \times \{\frac{1}{2}\} \times \{\frac{1}{2}\}, \exists$ UMP since (*) obtains. As in probs 3,4, if $\Theta_0 = \{\frac{1}{2}\} \times \{\frac{1}{2}\}^c, \Theta_1 = \{\frac{1}{2}\}^c \times \{\frac{1}{2}\}$, $\nexists$ UMP. The test statistic in the first two cases is $N_X - N_Y$. item To compute p_F, let us consider the case $\Theta_0 = \{\frac{1}{2}\} \times (\frac{1}{2}, 1), \Theta_1 = (\frac{1}{2}, 1) \times \{\frac{1}{2}\}$. For the MPE criterion, $\eta(\alpha, \theta) = 1$ so that

$$\ell rt : N_X - N_Y \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\Rightarrow p_F(d^*) \triangleq \sup_{\theta \in (\frac{1}{2}, 1)} p_F(d^*_\theta)$$

i.e.,

$$\begin{aligned}
p_F(d^*) &= \sup_{\theta \in (\frac{1}{2},1)} P(N_X \geq N_Y | H = 0) \\
&= \sup_{\theta \in (\frac{1}{2},1)} \sum_{k=0}^{N} P(N_X \geq k | N_Y = k, H = 0) P(N_Y = k | H = 0) \\
&= \sup_{\theta \in (\frac{1}{2},1)} \sum_{k=0}^{N} P(N_X \geq k | H = 0) P(N_Y = k | H = 0)
\end{aligned}$$

recall : N_X is independent of N_Y under H_0 and H_1).

$$= \sup_{\theta \in (\frac{1}{2},1)} \sum_{k=0}^{N} \left(\sum_{i=k}^{N} \binom{N}{i} \left(\frac{1}{2}\right)^N \right) \left(\binom{N}{k} \theta^k (1-\theta)^{N-k} \right)$$

ENJOY IT!