ENEE 621: Estimation and Detection Theory

Problem Set 5 **Spring 2007**

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1. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. $\{0,1\}$-valued r.v.'s with $P(X_n = 0) = P_0 \quad \forall n$ under hypothesis H_0, and $P(X_n = 0) = \frac{1}{2} \quad \forall n$ under hypothesis H_1. Here, P_0 is a r.v. with values in the set $\{\frac{1}{4}, \frac{3}{4}\}$, and satisfying $P(P_0 = 1/4) = 1/4$. The observations consist of a sequence $\{Y_n\}_{n=1}^{N}$ of r.v.'s where $Y_1 = 0$ a.s., and

$$Y_{n+1} = Y_n \oplus X_n, n = 1, 2, \ldots, N-1,$$

with $\oplus$ denoting addition modulo 2.

(a) Determine a likelihood ratio test (LRT) for testing H_0 vs. H_1. What is the test statistic?

(b) Under H_0, determine the MAP estimate of P_0 given $Y_1, \ldots, Y_n$ in terms of the test statistic of part (a).

2. Consider the following hypothesis testing problem:

$$H_0 : Y \sim f_0(y) = \begin{cases} 1 - |y|, & |y| \leq 1 \\ 0, & |y| > 1, \end{cases}$$

$$H_1 : Y \sim f_1(y) = \begin{cases} \frac{2-|y|}{4}, & |y| \leq 2 \\ 0, & |y| > 2. \end{cases}$$

Given that deciding H_0 when H_1 is true costs twice as much as deciding H_1 when H_0 is true, and that correct decisions cost nothing, and further that $P(H = 1) = p, p\epsilon(0, 1)$, find the Bayes' decision rule as a function of p.

3. Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. $\{0,1\}$-valued r.v.'s with $P(Z_n = 0) = p_0 \quad \forall n$ under H_0, and $P(Z_n = 0) = p_1 \quad \forall n$ under H_1. Assume that $0 < p_0, p_1 < 1,\ p_0 \neq p_1,\ p_0$ is known, and p_1 is an unknown constant. The observations consist of a sequence $\{Y_n\}_{n=1}^{N}$ of r.v.'s, where

$$Y_1 = 0 \text{ a.s.}, \quad Y_{n+1} = Y_n \oplus Z_n,\ n = 1, 2, \ldots, (N-1).$$

with $\oplus$ denoting addition modulo 2.

(a) Construct a LRT for testing H_0 vs. H_1, and determine the test statistic.

(b) What are the requirements for p_0 and p_1 for a UMP test to exist?

4. Consider the following binary hypothesis testing problem. Under $H_0, Y \sim$ Rayleigh with **known** parameter $\sigma_0^2 (> 0)$, whereas under H_1, $Y \sim$ Rayleigh with **unknown** (deterministic) parameter $\sigma_1^2 (> 0)$ lying in the set $I \subseteq (0, \infty)$.

   Is there a UMP test for H_0 vs. H_1 when

   (a) $I = (\sigma_0^2, +\infty)$

   (b) $I = (0, \sigma_0^2) \bigcup (\sigma_0^2, +\infty)$

   (c) $I = (0, \sigma_0^2)$

5. Consider the following hypotheses:

$$H_0 : \text{ coin is fair}, P(\text{head}) = 1/2;$$

$$H_1 : \text{ coin is biased}, \ P(\text{head}) = p, \ p\epsilon(1/2, 1).$$

   A decision has to be made on the basis of N independent tosses of the coin. For $1 \leq N < \infty$, find the LRT and identify the statistic S_N. Find the probability distribution of S_N under each hypothesis. For a Neyman-Pearson test with $p_F \leq \alpha$, find the smallest number of trials (i.e., coin tosses).

6. Consider the following hypothesis testing problem:

$$H_0 : Y_i = N_i, \ i = 1, \ldots, K,$$

$$H_1 : Y_i = s_i + N_i, \ i = 1, \ldots, K,$$

   where $\{N_i\}_{i=1}^K$ is a sequence of zero mean $\mathbb{R}$-valued Gaussian r.v.'s with $E[N_i N_j] = \min\{i, j\}$. For a given $p_F \leq \alpha$, find the Neyman-Pearson test for the following cases:

   (a) $s_i \equiv 1, \quad i = 1, \ldots, K;$

   (b) $s_i = i, \quad i = 1, \ldots, K.$

   Calculate p_D in each case.

7. It is known that one of two coins is fair and that the other has a probability θ of coming up heads when flipped – here θ is unknown and deterministic, $\theta \in (0, 1), \theta \neq 1/2$.

The coins are tossed simultaneously and independently to produce an independent sequence of samples $(X_1, Y_1), \ldots, (X_N, Y_N)$, where X_i and Y_i are each 0 or 1 according to whether the respective coin toss is a tail or a head. You are required to identify the fair coin.

(a) Formulate the problem as one of binary hypothesis testing and determine the generalized likelihood ratio test. What is the test statistic?

(b) Determine the condition under which the test is UMP. What is the test statistic now?

(c) In part (b), assuming the MPE criterion, what is the probability of a false alarm?