ENEE 621
SPRING 2016
ESTIMATION AND DETECTION THEORY

ANSWER KEY TO TEST # 1:

1. ______

1.a. For each $h = 0, 1$, the probability distribution F_h has probability density function $f_h : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_h(y) = \begin{cases} 0 & \text{if } y < 0 \\ \alpha_h e^{-\alpha_h y} & \text{if } y \geq 0. \end{cases}$$

Therefore,

$$\begin{aligned} d_\eta(y) = 0 \quad & \text{iff} \quad f_1(y) < \eta f_0(y) \\ & \text{iff} \quad \alpha_1 e^{-\alpha_1 y} < \eta \alpha_0 e^{-\alpha_0 y}, \ y \geq 0 \\ & \text{iff} \quad e^{-(\alpha_1 - \alpha_0) y} < \eta \frac{\alpha_0}{\alpha_1}, \ y \geq 0 \\ & \text{iff} \quad \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < (\alpha_1 - \alpha_0) y, \ y \geq 0 \\ & \text{iff} \quad \frac{1}{\alpha_1 - \alpha_0} \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < y, \ y \geq 0 \end{aligned} \tag{1.1}$$

It is plain that

$$C(d_\eta) = \left\{ y \geq 0 : \frac{1}{\alpha_1 - \alpha_0} \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < y \right\} \tag{1.2}$$

1.b. Obviously,

$$\begin{aligned} P_F(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 0\right] \\ &= 1 - \mathbb{P}\left[d_\eta(Y) = 0 | H = 0\right] \\ &= 1 - \mathbb{P}\left[\frac{1}{\alpha_1 - \alpha_0} \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < Y | H = 0\right] \\ &= 1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0}\left(\log\left(\frac{\alpha_1}{\eta \alpha_0}\right)\right)^+} \end{aligned} \tag{1.3}$$

In a similar way, we get

$$\begin{aligned} P_D(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 1\right] \\ &= 1 - \mathbb{P}\left[d_\eta(Y) = 0 | H = 1\right] \\ &= 1 - \mathbb{P}\left[\frac{1}{\alpha_1 - \alpha_0}\log\left(\frac{\alpha_1}{\eta\alpha_0}\right) < Y | H = 1\right] \\ &= 1 - e^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)\right)^+} \end{aligned} \tag{1.4}$$

1.c. With the notation introduced in the Lecture Notes we have

$$V(p) = J_p(d^\star(p)) = J_p(d_{\eta(p)}), \quad p \in [0,1]$$

where

$$\eta(p) = \frac{\Gamma_0(1-p)}{\Gamma_1 p} = \frac{1-p}{p}$$

since here $\Gamma_0 = \Gamma_1 = 1$. It is now straightforward to see that

$$\begin{aligned} V(p) &= p\mathbb{P}\left[d_{\eta(p)}(Y) = 1 | H = 1\right] + (1-p)\mathbb{P}\left[d_{\eta(p)}(Y) = 0 | H = 0\right] \\ &= pP_D(d_{\eta(p)}) + (1-p)\left(1 - P_F(d_{\eta(p)})\right) \\ &= p\left(1 - e^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)^+}\right) + (1-p)e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)^+} \\ &= \begin{cases} p\left(1 - e^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)}\right) + (1-p)e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)} & \text{if } \eta(p)\alpha_0 < \alpha_1 \\ 1-p & \text{if } \alpha_1 \le \eta(p)\alpha_0 \end{cases} \\ &= \begin{cases} p\left(1 - \left(\frac{\eta(p)\alpha_0}{\alpha_1}\right)^{-\frac{\alpha_1}{\alpha_1-\alpha_0}}\right) + (1-p)\left(\frac{\eta(p)\alpha_0}{\alpha_1}\right)^{\frac{\alpha_0}{\alpha_1-\alpha_0}} & \text{if } \eta(p)\alpha_0 < \alpha_1 \\ 1-p & \text{if } \alpha_1 \le \eta(p)\alpha_0 \end{cases} \end{aligned} \tag{1.5}$$

1.d. Fix P_F in $[0,1]$ and solve the equation

$$P_F(d_\eta) = P_F, \quad \eta \ge 0$$

Thus,

$$1 - e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)\right)^+} = P_F$$

or equivalently,

$$1 - P_F = e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)\right)^+}$$

or equivalently,

$$-\log\left(1 - P_F\right) = \frac{\alpha_0}{\alpha_1 - \alpha_0}\left(\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)\right)^+$$

2. ____

2.a.
2.b.

3.

3.a.
3.c.

4.

4.a.
4.b.
