ENEE 621
SPRING 2016
ESTIMATION AND DETECTION THEORY

ANSWER KEY TO TEST # 1:

1. ____

1.a. For each $h = 0, 1$, the probability distribution F_h has probability density function $f_h : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_h(y) = \begin{cases} 0 & \text{if } y < 0 \\ \\ \alpha_h e^{-\alpha_h y} & \text{if } y \geq 0. \end{cases}$$

Therefore,

$$\begin{aligned} d_\eta(y) = 0 \quad & \text{iff} \quad f_1(y) < \eta f_0(y) \\ & \text{iff} \quad \alpha_1 e^{-\alpha_1 y} < \eta \alpha_0 e^{-\alpha_0 y}, \ y \geq 0 \\ & \text{iff} \quad e^{-(\alpha_1 - \alpha_0) y} < \eta \frac{\alpha_0}{\alpha_1}, \ y \geq 0 \\ & \text{iff} \quad \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < (\alpha_1 - \alpha_0) y, \ y \geq 0 \\ & \text{iff} \quad \frac{1}{\alpha_1 - \alpha_0} \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < y, \ y \geq 0. \end{aligned} \tag{1.1}$$

It is plain that

$$C(d_\eta) = \left\{ y \geq 0 : \frac{1}{\alpha_1 - \alpha_0} \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < y \right\}. \tag{1.2}$$

1.b. Obviously,

$$\begin{aligned} P_F(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 0\right] \\ &= 1 - \mathbb{P}\left[d_\eta(Y) = 0 | H = 0\right] \\ &= 1 - \mathbb{P}\left[\frac{1}{\alpha_1 - \alpha_0} \log\left(\frac{\alpha_1}{\eta \alpha_0}\right) < Y | H = 0\right] \\ &= 1 - e^{-\frac{\alpha_0}{\alpha_1 - \alpha_0}\left(\log\left(\frac{\alpha_1}{\eta \alpha_0}\right)\right)^+}. \end{aligned} \tag{1.3}$$

In a similar way, we get

$$\begin{aligned} P_D(d_\eta) &= \mathbb{P}\left[d_\eta(Y)=1|H=1\right] \\ &= 1-\mathbb{P}\left[d_\eta(Y)=0|H=1\right] \\ &= 1-\mathbb{P}\left[\frac{1}{\alpha_1-\alpha_0}\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)<Y|H=1\right] \\ &= 1-e^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)\right)^+}. \end{aligned}$$

1.c. With the notation introduced in the Lecture Notes we have

$$V(p)=J_p(d^\star(p))=J_p(d_{\eta(p)}),\quad p\in[0,1]$$

where

$$\eta(p)=\frac{\Gamma_0(1-p)}{\Gamma_1 p}=\frac{1-p}{p}$$

since here $\Gamma_0=\Gamma_1=1$. It is now straightforward to see that

$$\begin{aligned} V(p) &= p\mathbb{P}\left[d_{\eta(p)}(Y)=0|H=1\right]+(1-p)\mathbb{P}\left[d_{\eta(p)}(Y)=1|H=0\right] \\ &= p\left(1-P_D(d_{\eta(p)})\right)+(1-p)P_F(d_{\eta(p)}) \\ &= pe^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)^+}+(1-p)\left(1-e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)^+}\right) \\ &= \begin{cases} pe^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)}+(1-p)\left(1-e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(p)\alpha_0}\right)\right)}\right) & \text{if } \eta(p)\alpha_0<\alpha_1 \\ p & \text{if } \alpha_1\le\eta(p)\alpha_0 \end{cases} \\ &= \begin{cases} p\left(\frac{\eta(p)\alpha_0}{\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1-\alpha_0}}+(1-p)\left(1-\left(\frac{\eta(p)\alpha_0}{\alpha_1}\right)^{\frac{\alpha_0}{\alpha_1-\alpha_0}}\right) & \text{if } \eta(p)\alpha_0<\alpha_1 \\ p & \text{if } \alpha_1\le\eta(p)\alpha_0. \end{cases} \\ &= \begin{cases} p\left(\frac{(1-p)\alpha_0}{p\alpha_1}\right)^{\frac{\alpha_1}{\alpha_1-\alpha_0}}+(1-p)\left(1-\left(\frac{(1-p)\alpha_0}{p\alpha_1}\right)^{\frac{\alpha_0}{\alpha_1-\alpha_0}}\right) & \text{if } \frac{\alpha_0}{\alpha_0+\alpha_1}<p\le 1 \\ p & \text{if } 0\le p\le\frac{\alpha_0}{\alpha_0+\alpha_1}. \end{cases} \end{aligned}$$

1.d. First note that

$$\begin{aligned} P_F(d_\eta) &= 1-e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)\right)^+} \\ &= \begin{cases} 1-e^{-\frac{\alpha_0}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta\alpha_0}\right)\right)} & \text{if } \eta<\frac{\alpha_1}{\alpha_0} \\ 0 & \text{if } \frac{\alpha_1}{\alpha_0}\le\eta \end{cases} \\ &= \begin{cases} 1-\left(\frac{\eta\alpha_0}{\alpha_1}\right)^{\frac{\alpha_0}{\alpha_1-\alpha_0}} & \text{if } \eta<\frac{\alpha_1}{\alpha_0} \\ 0 & \text{if } \frac{\alpha_1}{\alpha_0}\le\eta \end{cases} \end{aligned}$$

Fix P_F in $(0,1]$ and solve the equation

$$P_F(d_\eta) = P_F, \quad \eta \geq 0.$$

In view of the previous calculations, this amounts to solving

$$\left(\frac{\eta\alpha_0}{\alpha_1}\right)^{\frac{\alpha_0}{\alpha_1-\alpha_0}} = 1 - P_F, \quad 0 \leq \eta < \frac{\alpha_1}{\alpha_0}.$$

This has a unique solution $\eta(P_F)$ given by

$$\eta(P_F) = \frac{\alpha_1}{\alpha_0}\left(1-P_F\right)^{\frac{\alpha_1-\alpha_0}{\alpha_0}}.$$

The corresponding point P_D on the ROC curve is therefore given by $P_D(d_{\eta(P_F)})$ evaluated as

$$P_D(d_{\eta(P_F)}) = 1 - e^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(P_F)\alpha_0}\right)\right)^+}. \tag{1.4}$$

But

$$\frac{\alpha_1}{\eta(P_F)\alpha_0} = \frac{\alpha_1}{\alpha_0 \cdot \frac{\alpha_1}{\alpha_0}\left(1-P_F\right)^{\frac{\alpha_1-\alpha_0}{\alpha_0}}} = \left(1-P_F\right)^{-\frac{\alpha_1-\alpha_0}{\alpha_0}} > 1$$

and

$$\log\left(\frac{\alpha_1}{\eta(P_F)\alpha_0}\right) = \log\left(1-P_F\right)^{-\frac{\alpha_1-\alpha_0}{\alpha_0}} = -\frac{\alpha_1-\alpha_0}{\alpha_0}\log\left(1-P_F\right) > 0.$$

Therefore,

$$\begin{aligned} P_D(d_{\eta(P_F)}) &= 1 - e^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\left(\log\left(\frac{\alpha_1}{\eta(P_F)\alpha_0}\right)\right)} \\ &= 1 - e^{-\frac{\alpha_1}{\alpha_1-\alpha_0}\cdot\left(-\frac{\alpha_1-\alpha_0}{\alpha_0}\log(1-P_F)\right)} \\ &= 1 - e^{\frac{\alpha_1}{\alpha_0}\log(1-P_F)} \\ &= 1 - (1-P_F)^{\frac{\alpha_1}{\alpha_0}}. \end{aligned} \tag{1.5}$$

We conclude that $\Gamma : [0,1] \to [0,1]$ is given by

$$\Gamma(P_F) = 1 - (1-P_F)^{\frac{\alpha_1}{\alpha_0}}, \quad P_F \in [0,1].$$

2. ____

Recall that a rv Y is said to be Rayleigh distributed with parameter $\theta > 0$ if its probability distribution F_θ admits a probability density function $f_\theta : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_\theta(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{y}{\theta^2}e^{-\frac{y^2}{2\theta^2}} & \text{if } y \geq 0. \end{cases}$$

It is crucial to observe that

$$\begin{aligned} F_\theta(y) &= \int_{-\infty}^{y} f_\theta(x)dx \\ &= 1 - e^{-\frac{(y^+)^2}{2\theta^2}}, \quad y \in \mathbb{R}. \end{aligned} \tag{1.6}$$

In particular, for each t in $\mathbb{R}$, we have

$$\mathbb{P}_\theta\left[Y^2 > t\right] = e^{-\frac{t^+}{2\theta^2}}.$$

2.a. With distinct θ_0 and θ_1 in $(0,\infty)$, consider the binary hypothesis testing problem

$$\begin{array}{ll} H_1: & Y \sim F_{\theta_1} \\ H_0: & Y \sim F_{\theta_0}. \end{array} \tag{1.7}$$

For $\eta > 0$, consider the corresponding test $d_\eta : \mathbb{R} \to \{0,1\}$. In a routine manner we find

$$\begin{aligned} d_\eta(y) = 0 \quad & \text{iff} \quad f_{\theta_1}(y) < \eta f_{\theta_0}(y) \\ & \text{iff} \quad \frac{y}{\theta_1^2} e^{-\frac{y^2}{2\theta_1^2}} < \eta \frac{y}{\theta_0^2} e^{-\frac{y^2}{2\theta_0^2}}, \ y > 0 \\ & \text{iff} \quad e^{-\frac{y^2}{2}\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)} < \eta \frac{\theta_1^2}{\theta_0^2}, \ y > 0 \\ & \text{iff} \quad e^{-\frac{y^2}{2} D(\theta_1,\theta_0)} < \eta R(\theta_1,\theta_0), \ y > 0 \end{aligned}$$

with

$$D(\theta_1,\theta_0) = \frac{1}{\theta_1^2} - \frac{1}{\theta_0^2} \quad \text{and} \quad R(\theta_1,\theta_0) = \frac{\theta_1^2}{\theta_0^2}.$$

Taking logarithms on both sides, we get

$$d_\eta(y) = 0 \quad \text{iff} \quad -2\log\left(\eta R(\theta_1,\theta_0)\right) < D(\theta_1,\theta_0) y^2, \ y > 0. \tag{1.8}$$

It follows that

$$\begin{aligned} P_F(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 0\right] \\ &= 1 - \mathbb{P}\left[d_\eta(Y) = 0 | H = 0\right] \\ &= 1 - \mathbb{P}\left[-2\log\left(\eta R(\theta_1,\theta_0)\right) < D(\theta_1,\theta_0) Y^2 | H = 0\right]. \end{aligned} \tag{1.9}$$

If $0 < \theta_0 < \theta_1$, then $D(\theta_1,\theta_0) < 0$ and $R(\theta_1,\theta_0) > 1$, so that

$$\begin{aligned} P_F(d_\eta) &= 1 - \mathbb{P}\left[Y^2 < -\frac{2\log\left(\eta R(\theta_1,\theta_0)\right)}{D(\theta_1,\theta_0)} \Big| H = 0\right] \\ &= \mathbb{P}\left[Y^2 \geq -\frac{2\log\left(\eta R(\theta_1,\theta_0)\right)}{D(\theta_1,\theta_0)} \Big| H = 0\right] \\ &= e^{-\frac{1}{2\theta_0^2}\cdot\left(-\frac{2\log(\eta R(\theta_1,\theta_0))}{D(\theta_1,\theta_0)}\right)^+}. \end{aligned} \tag{1.10}$$

With α in $(0,1)$, solving the equation

$$e^{-\frac{1}{2\theta_0^2}\cdot\left(-\frac{2\log(\eta R(\theta_1,\theta_0))}{D(\theta_1,\theta_0)}\right)^+} = \alpha, \quad \eta > 0 \tag{1.11}$$

requires

$$\log\left(\eta R(\theta_1,\theta_0)\right) > 0,$$

or equivalently,

$$\eta R(\theta_1,\theta_0) > 1.$$

Under that condition, we get

$$e^{-\frac{1}{2\theta_0^2}\cdot\left(-\frac{2\log(\eta R(\theta_1,\theta_0))}{D(\theta_1,\theta_0)}\right)^+} = e^{\frac{\log(\eta R(\theta_1,\theta_0))}{\theta_0^2 D(\theta_1,\theta_0)}} \tag{1.12}$$

and the equation (1.11) becomes

$$\log\left(\eta R(\theta_1,\theta_0)\right) = \theta_0^2 D(\theta_1,\theta_0)\cdot\log\alpha.$$

The solution $\eta(\alpha)$ satisfies

$$\eta(\alpha)R(\theta_1,\theta_0) = \alpha^{\theta_0^2 D(\theta_1,\theta_0)},$$

and is therefore given by

$$\eta(\alpha) = \frac{\alpha^{\theta_0^2 D(\theta_1,\theta_0)}}{R(\theta_1,\theta_0)}.$$

The Neyman-Pearson test $d_{\rm NP}(\theta_1,\theta_0;\alpha)$ of size α is characterized by

$$\begin{aligned} d_{\rm NP}(\theta_1,\theta_0;\alpha)(y) = 0 \quad & \text{iff} \quad -2\log\left(\eta(\alpha)R(\theta_1,\theta_0)\right) < D(\theta_1,\theta_0)y^2,\ y > 0 \\ & \text{iff} \quad -2\theta_0^2 D(\theta_1,\theta_0)\cdot\log\alpha < D(\theta_1,\theta_0)y^2,\ y > 0 \\ & \text{iff} \quad 2\theta_0^2\cdot\log\alpha < -y^2,\ y \geq 0 \\ & \text{iff} \quad y^2 < -2\theta_0^2\cdot\log\alpha,\ y > 0. \end{aligned} \tag{1.13}$$

Note that

$$C(d_{\rm NP}(\theta_1,\theta_0;\alpha)) = \left\{y > 0:\ y^2 < -2\theta_0^2\cdot\log\alpha\right\}.$$

On the other hand, if $0 < \theta_1 < \theta_0$, then $D(\theta_1,\theta_0) > 0$ and $R(\theta_1,\theta_0) < 1$, so that

$$\begin{aligned} P_F(d_\eta) &= 1 - \mathbb{P}\left[Y^2 > -\frac{2\log\left(\eta R(\theta_1,\theta_0)\right)}{D(\theta_1,\theta_0)}\Big| H = 0\right] \\ &= 1 - e^{-\frac{1}{2\theta_0^2}\cdot\left(-\frac{2\log(\eta R(\theta_1,\theta_0))}{D(\theta_1,\theta_0)}\right)^+}. \end{aligned} \tag{1.14}$$

With α in $(0,1)$, solving the equation

$$1 - e^{-\frac{1}{2\theta_0^2}\cdot\left(-\frac{2\log(\eta R(\theta_1,\theta_0))}{D(\theta_1,\theta_0)}\right)^+} = \alpha, \quad \eta > 0 \tag{1.15}$$

requires

$$\log\left(\eta R(\theta_1,\theta_0)\right) < 0,$$

or equivalently,

$$\eta R(\theta_1, \theta_0) < 1.$$

Under that condition, we get

$$1 - e^{-\frac{1}{2\theta_0^2}\cdot\left(-2\frac{\log(\eta R(\theta_1,\theta_0))}{D(\theta_1,\theta_0)}\right)^+} = 1 - e^{\frac{\log(\eta R(\theta_1,\theta_0))}{\theta_0^2 D(\theta_1,\theta_0)}} \tag{1.16}$$

and the equation (1.15) becomes

$$\log\left(\eta R(\theta_1, \theta_0)\right) = \theta_0^2 D(\theta_1, \theta_0) \cdot \log(1-\alpha).$$

This yields

$$\eta R(\theta_1, \theta_0) = (1-\alpha)^{\theta_0^2 D(\theta_1,\theta_0)},$$

and the solution $\eta(\alpha)$ is given by

$$\eta(\alpha) = \frac{(1-\alpha)^{\theta_0^2 D(\theta_1,\theta_0)}}{R(\theta_1, \theta_0)}.$$

The Neyman-Pearson test $d_{\text{NP}}(\theta_1, \theta_0; \alpha)$ of size α is now characterized by

$$\begin{aligned} d_{\text{NP}}(\theta_1, \theta_0; \alpha)(y) = 0 \quad & \text{iff} \quad -2\log\left(\eta(\alpha) R(\theta_1, \theta_0)\right) < D(\theta_1, \theta_0) y^2, \ y \geq 0 \\ & \text{iff} \quad -2\theta_0^2 D(\theta_1, \theta_0) \cdot \log(1-\alpha) < D(\theta_1, \theta_0) y^2, \ y \geq 0 \\ & \text{iff} \quad -2\theta_0^2 \cdot \log(1-\alpha) < y^2, \ y \geq 0. \end{aligned} \tag{1.17}$$

Note that

$$C(d_{\text{NP}}(\theta_1, \theta_0; \alpha)) = \left\{ y \geq 0 : \ -\theta_0^2 \cdot \log(1-\alpha) < y^2 \right\}.$$

2.b. With $\Theta_0 = \{1\}$ and $\theta_1 = (1, \infty)$, it is plain that there exists a UMP test of size α. Indeed note that

$$C(d_{\text{NP}}(\theta_1, 1; \alpha)) = \left\{ y > 0 : \ y^2 < -2\log\alpha \right\}, \quad \theta_1 > 1.$$

These tests are all Neyman-Pearson tests of size α implementing the *same* decision regions without having to require explicit knowledge of θ_1. All that is needed is that $\theta_1 > 1$!
2.c. When $\Theta_0 = (0, 1)$ and $\theta_1 = (1, \infty)$, there is no UMP test.

3. ____

3.a. In Chapter 3 we have seen that when all the hypotheses are equally likely, namely

$$p_0 = \ldots = p_{M-1} = \frac{1}{M},$$

the optimal test under the probability of error criterion is the Maximum Likelihood test $d_{\text{ML}} : \mathbb{R} \to \{0, 1, \ldots, M-1\}$ given by

$$d_{\text{ML}}(y) = \arg\max\left(\ell = 0, \ldots, M-1 : \ f_{\theta_\ell}(y)\right), \quad y \in \mathbb{R}$$

with a lexicographic tiebreaker in the event of ties. In other words,

$$d_{\mathrm{ML}}(y) = m \quad \text{iff} \quad f_{\theta_m}(y) = \max\left(f_{\theta_\ell}(y),\ \ell = 0, 1, \ldots, M-1\right)$$

with a lexicographic tiebreaker in the event of ties.

However, we note that

$$\begin{aligned}
&\max\left(f_{\theta_\ell}(y),\ \ell = 0, 1, \ldots, M-1\right) \\
&\quad = \max\left(g(y-\theta_\ell),\ \ell = 0, 1, \ldots, M-1\right) \\
&\quad = \max\left(g(|y-\theta_\ell|),\ \ell = 0, 1, \ldots, M-1\right) \quad \text{[By symmetry]} \\
&\quad = g(\min\left(|y-\theta_\ell|,\ \ell = 0, 1, \ldots, M-1\right)) \quad \text{[By strict decreasing monotonicity on } \mathbb{R}_+\text{]}.
\end{aligned}$$

This implies that

$$d_{\mathrm{ML}}(y) = m \quad \text{iff} \quad |y-\theta_m| = \min\left(|y-\theta_\ell|,\ \ell = 0, 1, \ldots, M-1\right)$$

with a lexicographic tiebreaker in the event of ties. The geometric interpretation is clear: Given the observation y, the test d_{ML} selects that hypothesis H_m whose parameter θ_m is closest to y – This is sometimes known as the *nearest neighbor* detector.

It is plain that the nearest neighbor detector depends on $g : \mathbb{R} \to \mathbb{R}_+$ only through conditions (i)–(iii), not on the specific form of $g : \mathbb{R} \to \mathbb{R}_+$. For instance, the two densities

$$g(y) = \frac{\alpha}{2} e^{-\alpha|y|}, \quad y \in \mathbb{R}$$

and

$$g(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}$$

will yield the same conclusion.

3.b. As in the binary case, randomization does not affect optimality in the M-ary case – This was not done in the Lecture Notes but can be easily shown by similar arguments. In particulate the ML test d_{ML} is also optimal among all admissible randomized tests. Therefore, by the optimality of d_{ML} we must have

$$\mathbb{P}\left[d_{\mathrm{ML}}(Y) \neq H\right] < \mathbb{P}\left[D_R \neq H\right]$$

where D_R is the decision to flip an M-sided coin (independently of everything else) with

$$\mathbb{P}\left[D_R = m\right] = \frac{1}{M}, \quad m = 0, \ldots, M-1.$$

But, assuming an arbitrary pdf $\boldsymbol{p}$ for the rv H, we see that

$$\begin{aligned}
\mathbb{P}\left[D_R \neq H\right] &= \sum_{m=0}^{M-1} \mathbb{P}\left[H \neq m, D_R = m\right] \\
&= \sum_{m=0}^{M-1} \mathbb{P}\left[H \neq m\right] \mathbb{P}\left[D_R = m\right] \\
&= \sum_{m=0}^{M-1} (1 - p_m) \frac{1}{M} \\
&= \frac{1}{M} \sum_{m=0}^{M-1} (1 - p_m) \\
&= \frac{M-1}{M}
\end{aligned}$$

since

$$\sum_{m=0}^{M-1} (1 - p_m) = M - \sum_{m=0}^{M-1} p_m = M - 1.$$

The result of this calculation is independent of the prior $\boldsymbol{p}$ on H.
