ENEE 621
SPRING 2016
DETECTION AND ESTIMATION THEORY

ANSWER KEY TO FINAL EXAM:

1.

Fact: For any Borel mapping $u : \mathbb{R} \to \mathbb{R}$ which is integrable, i.e.,

$$\int_{\mathbb{R}} |u(y)| dy < \infty$$

we necessarily have

$$\lim_{y \to \pm\infty} |u(y)| = 0.$$

1.a. The point of this question was to extract the conditions on the probability density function $h : \mathbb{R} \to \mathbb{R}_+$ which ensure that the conditions **(CR1)-(CR5)** hold for the family $\{f_\theta, \ \theta \in \mathbb{R}\}$. They are

(i) The support of $h : \mathbb{R} \to \mathbb{R}_+$ is the entirely line $\mathbb{R}$, namely

$$h(y) > 0, \quad y \in \mathbb{R}. \tag{1.1}$$

(ii) The probability density function $h : \mathbb{R} \to \mathbb{R}_+$ is differentiable everywhere on $\mathbb{R}$.

(iii) The square-integrability condition

$$\int_{\mathbb{R}} \left| \frac{h'(y)}{h(y)} \right|^2 h(y) dy < \infty \tag{1.2}$$

holds.

(iv) The derivative $h' : \mathbb{R} \to \mathbb{R}$ is integrable in that

$$\int_{\mathbb{R}} |h'(y)| \, dy < \infty. \tag{1.3}$$

(CR1) This condition is automatically satisfied here since $\Theta = (0, \infty)$ *is* an open set in $\mathbb{R}$.

(CR2) For each θ in $\mathbb{R}$, F_θ is absolutely continuous with respect to Lebesgue measure with probability density function $f_\theta : \mathbb{R} \to \mathbb{R}_+$ given by

$$f_\theta(y) = h(y - \theta), \quad y \in \mathbb{R}$$

and **(CR2a)** holds. The support $S(\theta)$ of the probability density function $f_\theta : \mathbb{R} \to \mathbb{R}_+$ is given by

$$S(\theta) = \{y \in \mathbb{R} : \ f_\theta(y) > 0\} = \{y \in \mathbb{R} : \ h(y - \theta) > 0\}$$

so that

$$S(\theta) = \theta + S(0), \quad \theta \in \mathbb{R}$$

It follows that we must have $S(\theta) = \mathbb{R}$ for each θ in $\mathbb{R}$ and **(CR2b)** holds with $S = \mathbb{R}$. This obviously requires that (1.1) holds.

(CR3) Assuming the existence of needed derivatives we get

$$\frac{\partial}{\partial \theta} f_\theta(y) = -h'(y - \theta), \quad \begin{array}{c} \theta \in \mathbb{R} \\ y \in \mathbb{R} \end{array}$$

where $h' : \mathbb{R} \to \mathbb{R}$ is the derivative of h. Thus, **(CR3)** requires that the probability density function $h : \mathbb{R} \to \mathbb{R}_+$ be differentiable *everywhere* on $\mathbb{R}$.

(CR4) This integrability condition reads

$$\mathbb{E}_\theta \left[\left| \frac{\partial}{\partial \theta} \log f_\theta(Y) \right|^2 \right] = \int_{\mathbb{R}} \left| \frac{h'(y - \theta)}{h(y - \theta)} \right|^2 h(y - \theta) dy < \infty, \quad \theta \in \mathbb{R},$$

and reduces to the *single* integrability condition (1.2).

(CR5) This regularity condition amounts to

$$0 = \int_{\mathbb{R}} \frac{\partial}{\partial \theta} f_\theta(y) dy, \quad \theta \in \mathbb{R}$$

i.e.,

$$0 = \int_{\mathbb{R}} h'(y - \theta) dy, \quad \theta \in \mathbb{R}$$

Thus, by a simple of variable we see that **(CR5)** holds provided the integrability condition (1.3) holds for the derivative $h' : \mathbb{R} \to \mathbb{R}$ with

$$0 = \int_{\mathbb{R}} h'(y) dy.$$

Note that

$$\int_{\mathbb{R}} h'(y) dy = \lim_{A,B \to \infty} \int_{-A}^{B} h'(y) dy = \lim_{A,B \to \infty} (h(B) - h(-A)) = 0$$

and this always holds by virtue of the assumed integrability (1.3).

1.b. It is plain that the Fisher information matrix is given

$$\begin{aligned} M(\theta) &= \mathbb{E}_\theta\left[\left|\frac{\partial}{\partial\theta}\log f_\theta(Y)\right|^2\right] \\ &= \int_{\mathbb{R}} \left(\frac{h'(y-\theta)}{h(y-\theta)}\right)^2 h(y-\theta)dy \\ &= \int_{\mathbb{R}} \left(\frac{h'(z)}{h(z)}\right)^2 h(z)dz, \quad \theta\in\mathbb{R}. \end{aligned} \tag{1.4}$$

This quantity does not depend on θ.

1.c. Fix θ in $\mathbb{R}$. Under the integrability condition on $h:\mathbb{R}\to\mathbb{R}_+$, namely

$$\int_{\mathbb{R}} |z|h(z)dz < \infty \tag{1.5}$$

the integral

$$\mu_h = \int_{\mathbb{R}} zh(z)dz$$

is well defined and finite – In fact μ_h is simply the first moment under the probability density function $h:\mathbb{R}\to\mathbb{R}_+$. Hence, the estimator $g:\mathbb{R}\to\mathbb{R}$ is a finite mean estimator since

$$\begin{aligned} \mathbb{E}_\theta[g(Y)] &= \int_{\mathbb{R}} g(y)f_\theta(y)dy \\ &= \int_{\mathbb{R}} (ay+b)h(y-\theta)dy \\ &= \int_{\mathbb{R}} (az+(a\theta+b))h(z)dz \quad [z=y-\theta] \\ &= a\mu_h + (a\theta+b) \end{aligned} \tag{1.6}$$

It follows that

$$\frac{d}{d\theta}\mathbb{E}_\theta[g(Y)] = a.$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}} g(y)\frac{\partial}{\partial\theta}f_\theta(y)dy &= a\int_{\mathbb{R}} y\frac{\partial}{\partial\theta}f_\theta(y)dy + b\int_{\mathbb{R}}\frac{\partial}{\partial\theta}f_\theta(y)dy \\ &= a\int_{\mathbb{R}} y\frac{\partial}{\partial\theta}f_\theta(y)dy \end{aligned} \tag{1.7}$$

since

$$\int_{\mathbb{R}}\frac{\partial}{\partial\theta}f_\theta(y)dy = 0$$

by the regularity condition **(CR5)**. Next, we see that

$$\begin{aligned}\int_{\mathbb{R}} y\frac{\partial}{\partial\theta}f_\theta(y)dy &= \int_{\mathbb{R}} y(-h'(y-\theta))dy \\ &= -\int_{\mathbb{R}}(z+\theta)h'(z)dz \qquad (1.8)\end{aligned}$$

under the condition

$$\int_{\mathbb{R}} |z||h'(z)|dz < \infty.$$

Here as well we have

$$\int_{\mathbb{R}} h'(z)dz = \lim_{A,B\to\infty}\int_{-A}^{B} h'(z)dz = \lim_{A,B\to\infty}(h(B)-h(-A)) = 0$$

as before. On the other hand, integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}} zh'(z)dz &= \lim_{A,B\to\infty}\int_{-A}^{B} zh'(z)dz \\ &= \lim_{A,B\to\infty}\left([zh(z)]_{-A}^{B} - \int_{-A}^{B} h(z)dz\right) \\ &= \lim_{A,B\to\infty}(Bh(B)+Ah(-A)-(H(B)-H(-A))) = -1 \qquad (1.9)\end{aligned}$$

because $\lim_{z\to\infty}|z||h(z)| = 0$ as a result of the integrability condition (1.5); see Fact. Therefore,

$$\int_{\mathbb{R}} g(y)\frac{\partial}{\partial\theta}f_\theta(y)dy = a,$$

and the affine estimator is indeed regular under the integrability condition (1.5).

2.

2.a. For each $\theta > 0$,

$$\begin{aligned}1 &= \int_{\mathbb{R}} K(\theta)e^{-\frac{y^4}{\theta^4}}dy \\ &= \int_{\mathbb{R}} \theta K(\theta)e^{-\frac{y^4}{\theta^4}}\theta^{-1}dy \qquad (1.10) \\ &= \int_{\mathbb{R}} \theta K(\theta)e^{-z^4}dz \quad \left[z=\frac{y}{\theta}\right] \qquad (1.11) \\ &= \frac{\theta K(\theta)}{K(1)}\int_{\mathbb{R}} K(1)e^{-z^4}dz. \qquad (1.12)\end{aligned}$$

It follows that

$$1 = \frac{\theta K(\theta)}{K(1)},$$

whence

$$K(\theta) = \frac{K(1)}{\theta}.$$

2.b. Fix $n = 1, 2, \ldots$ and $\theta > 0$. For arbitrary $y_1, \ldots, y_n$ in $\mathbb{R}$, we have

$$\begin{aligned} f_\theta^{(n)}(y_1, \ldots, y_n) &= \prod_{i=1}^n K(\theta) e^{-\frac{y_i^4}{\theta^4}} \\ &= K(\theta)^n e^{-\frac{1}{\theta^4}\sum_{i=1}^n y_i^4} \\ &= K(1)^n \theta^{-n} e^{-\frac{1}{\theta^4}\sum_{i=1}^n y_i^4} \\ &= K(1)^n e^{-n\log\theta - \frac{1}{\theta^4}\sum_{i=1}^n y_i^4}. \end{aligned} \tag{1.13}$$

Thus,

$$\frac{\partial}{\partial\theta} \log f_\theta^{(n)}(y_1, \ldots, y_n) = -\frac{n}{\theta} + \frac{4}{\theta^5}\sum_{i=1}^n y_i^4.$$

The ML equation

$$\frac{\partial}{\partial\theta} \log f_\theta^{(n)}(y_1, \ldots, y_n) = 0, \quad \theta > 0$$

has a unique solution

$$g_{\mathrm{ML}}(y_1, \ldots, y_n) = \sqrt[4]{\frac{4}{n}\sum_{i=1}^n y_i^4}.$$

2.c. By the SLLNs for the rvs $\{Y_i^4,\ i = 1, 2, \ldots\}$ (under $\mathbb{P}_\theta$) it holds that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n Y_i^4 = \mathbb{E}_\theta\left[Y^4\right] \quad \mathbb{P}_\theta - a.s.,$$

whence

$$\lim_{n\to\infty} g_{\mathrm{ML}}(Y_1, \ldots, Y_n) = \sqrt[4]{4\mathbb{E}_\theta\left[Y^4\right]} \quad \mathbb{P}_\theta - a.s.$$

and it remains to evaluate $\mathbb{E}_\theta\left[Y^4\right]$.

We have

$$\begin{aligned} \mathbb{E}_\theta\left[Y^4\right] &= \int_{\mathbb{R}} K(\theta) y^4 e^{-\frac{y^4}{\theta^4}} dy \\ &= K(\theta)\theta^5 \int_{\mathbb{R}} \frac{y^4}{\theta^4} e^{-\frac{y^4}{\theta^4}} \theta^{-1} dy \\ &= K(\theta)\theta^5 \int_{\mathbb{R}} z^4 e^{-z^4} dz \quad \left[z = \frac{y}{\theta}\right] \qquad (1.14) \\ &= -\frac{K(\theta)}{4}\theta^5 \int_{\mathbb{R}} z \cdot (-4z^3 e^{-z^4}) dz \\ &= -\frac{K(\theta)}{4}\theta^5 \int_{\mathbb{R}} z \cdot \left(e^{-z^4}\right)' dz \qquad (1.15) \end{aligned}$$

with

$$\begin{aligned}\int_{\mathbb{R}} z \cdot \left(e^{-z^4}\right)' dz &= \left[ze^{-z^4}\right]_{-\infty}^{\infty} - \int_{\mathbb{R}} e^{-z^4} dz \\ &= -\int_{\mathbb{R}} e^{-z^4} dz \\ &= -K(1)^{-1} \end{aligned} \tag{1.16}$$

by integration by parts. Collecting we conclude that

$$\begin{aligned}\mathbb{E}_\theta\left[Y^4\right] &= -\frac{K(\theta)}{4}\theta^5\left(-K(1)^{-1}\right) \\ &= \frac{K(\theta)}{4}\theta^5 K(1)^{-1} \\ &= \frac{\theta^4}{4}\end{aligned} \tag{1.17}$$

as we use the fact $K(\theta) = K(1)\theta^{-1}$ established in Part **a**, so that

$$\sqrt[4]{4\mathbb{E}_\theta\left[Y^4\right]} = \theta.$$

The ML estimator is therefore strongly consistent.

3.

3.a. Here, for each $\theta > 0$,

$$f_\theta^{(3)}(y_1, y_2, y_3) = \frac{1}{8}e^{-\sum_{i=1}^{3}|y_i-\theta|}, \quad y_11, y_2, y_3 \in \mathbb{R}$$

To find the ML estimator $g_{\rm ML} : \mathbb{R}^3 \to (0, \infty)$ we proceed as follows: With observations y_1, y_2, y_3 given, we seek to find $g_{\rm ML}(y_1, y_2, y_3) > 0$ such that

$$\sum_{i=1}^{3} |y_i - g_{\rm ML}(y_1, y_2, y_3)| \leq \sum_{i=1}^{3} |y_i - \theta|, \quad \theta > 0.$$

Note that **here** $\theta > 0$, and **not** θ unconstrained in $\mathbb{R}$! For this unconstrained version of the problem we need to find $g_{\rm ML}(y_1, y_2, y_3)$ in $\mathbb{R}$ such that

$$\sum_{i=1}^{3} |y_i - g_{\rm ML}(y_1, y_2, y_3)| \leq \sum_{i=1}^{3} |y_i - \theta|, \quad \theta \in \mathbb{R}.$$

Its solution is well known and can be described as follows: Given the values y_1, y_2, y_3 in $\mathbb{R}$, write $y_{(1)}, y_{(2)}, y_{(3)}$ for these values ordered in increasing values (with a lexicographic tiebreaker), i.e., $\{y_1, y_2, y_3\} = \{y_{(1)}, y_{(2)}, y_{(3)}\}$ with

$$y_{(1)} \leq y_{(2)} \leq y_{(3)}.$$

Then, with this notation we have

$$g_{\rm ML}(y_1, y_2, y_3) = y_{(2)}.$$

It can be interpreted as the median for the uniform distribution on $\{y_1, y_2, y_3\}$!

To see why this is indeed true, observe that (i) $g_{\rm ML}(y_1, y_2, y_3)$ must necessarily lie in the interval $[y_{(1)}, y_{(3)}]$ – The metric of interest can always be decreased otherwise by moving towards the boundary points $y_{(1)}$ or $y_{(3)}$; and (ii) With $a < b$, we have

$$|a - \theta| + |b - \theta| = a - b, \quad \theta \in [a, b],$$

a fact which argues for the solution to necessarily be at $y_{(2)}$.

It is easy to see by a symmetry argument that

$$\mathbb{E}_\theta \left[Y_{(2)} \right] = \theta, \quad \theta \in \mathbb{R}.$$

Just take expectations in the identity

$$\sum_{i=1}^{3} Y_i = \sum_{i=1}^{3} Y_{(i)}$$

and use the fact that for each θ in $\mathbb{R}$, we have

$$\left(Y_{(1)} - \theta\right) =_{st} - \left(Y_{(3)} - \theta\right)$$

under $\mathbb{P}_\theta$.

However, here we need to solve the **constrained** problem: Find $g_{\rm ML}(y_1, y_2, y_3) > 0$ such that

$$\sum_{i=1}^{3} |y_i - g_{\rm ML}(y_1, y_2, y_3)| \le \sum_{i=1}^{3} |y_i - \theta|, \quad \theta > 0.$$

Four cases need to be considered:

(i) If $y_{(3)} \le 0$, then the ML estimate $g_{\rm ML}(y_1, y_2, y_3)$ does not exist (at least in the strict sense as an element of $(0, \infty)$). However one may decide to allow the search to be carried out over the larger set $\mathbb{R}_+$ (thereby including the boundary point $\theta = 0$), in which case $g_{\rm ML}(y_1, y_2, y_3) = 0$.

(ii) If $y_{(2)} \le 0 < y_{(3)}$, then $g_{\rm ML}(y_1, y_2, y_3) = 0$ (in the extended formulation, otherwise it does not exist).

(iii) If $y_{(1)} \le 0 < y_{(2)}$, then $g_{\rm ML}(y_1, y_2, y_3) = y_{(2)}$ (in the original formulation).

(iv) If $0 < y_{(1)}$, then $g_{\rm ML}(y_1, y_2, y_3) = y_{(2)}$ (in the original formulation).

3.b. The ML estimator cannot be an MVUE estimator since obviously

$$\mathbb{E}_\theta \left[g_{\rm ML}(Y_1, Y_2, Y_3) \right] \ne \theta > 0$$

by remarks above as we note that

$$Y_{(2)} \le g_{\rm ML}(Y_1, Y_2, Y_3).$$

3.c. The family $\{F_\theta^{(3)},\ \theta > 0\}$ is not an exponential family as can be checked by direct inspection (in spite of its "exponential nature").

4.

4.a. Fix $t \geq 0$ and $y = 0, 1, \ldots$. The posterior distribution of ϑ given $Y = y$ is easily computed as

$$f_{\vartheta|Y}(t|y) \quad = \quad \frac{\frac{t^y}{y!}e^{-t}g(t)}{\mathbb{P}\left[Y = y\right]}, \qquad \begin{array}{c} t \geq 0 \\ y = 0, 1, \ldots \end{array} \tag{1.18}$$

with

$$\mathbb{P}\left[Y = y\right] = \int_0^\infty \frac{\tau^y}{y!} e^{-\tau} g(\tau) d\tau, \quad y = 0, 1, \ldots$$

Therefore, for each $y = 0, 1, \ldots$, we get

$$\begin{aligned} \mathbb{E}\left[\vartheta|Y = y\right] &= \int_0^\infty t f_{\vartheta|Y}(t|y) dt \\ &= \frac{\int_0^\infty t \frac{t^y}{y!} e^{-t} g(t) dt}{\int_0^\infty \frac{t^y}{y!} e^{-t} g(t) dt} \\ &= \frac{W(y+1)}{W(y)} \end{aligned} \tag{1.19}$$

with

$$W(y) = \mathbb{E}\left[\vartheta^y e^{-\vartheta}\right] = \int_0^\infty t^y e^{-t} g(t) dt, \quad y = 0, 1, \ldots$$

4.b. Alternatively,

$$\mathbb{E}\left[\vartheta|Y = y\right] = (y+1)\frac{\mathbb{P}\left[Y = y+1\right]}{\mathbb{P}\left[Y = y\right]}, \quad y = 0, 1, \ldots$$

4.c. It is well known that

$$\widehat{\mathbb{E}}\left[\vartheta|Y = y\right] = \mu_\vartheta + \frac{\Sigma_{\vartheta Y}}{\Sigma_Y}\left(y - \mu_Y\right), \quad y \in \mathbb{R}.$$

Note that

$$\mathbb{E}\left[Y^p|\vartheta\right] = \begin{cases} \vartheta & \text{if } p = 1 \\ \vartheta^2 + \vartheta & \text{if } p = 2 \end{cases}$$

by standard properties of the Poisson distribution.

By standard preconditioning arguments it follows that

$$\mu_Y = \mathbb{E}\left[Y\right] = \mathbb{E}\left[\mathbb{E}\left[Y|\vartheta\right]\right] = \mathbb{E}\left[\vartheta\right], \tag{1.20}$$

and

$$\begin{aligned} \Sigma_Y &= \text{Var}[Y] \\ &= \mathbb{E}\left[Y^2\right] - \left(\mathbb{E}\left[Y\right]\right)^2 \\ &= \mathbb{E}\left[\mathbb{E}\left[Y^2|\vartheta\right]\right] - \left(\mathbb{E}\left[\vartheta\right]\right)^2 \\ &= \mathbb{E}\left[\vartheta^2 + \vartheta\right] - \left(\mathbb{E}\left[\vartheta\right]\right)^2 \\ &= \text{Var}\left[\vartheta\right] + \mathbb{E}\left[\vartheta\right]. \end{aligned} \tag{1.21}$$

In a similar vein, we have

$$\begin{aligned}\Sigma_{\vartheta Y} &= \text{Cov}\left[\vartheta, Y\right] \\ &= \mathbb{E}\left[\vartheta Y\right] - \mathbb{E}\left[\vartheta\right]\mathbb{E}\left[Y\right] \\ &= \mathbb{E}\left[\vartheta \mathbb{E}\left[Y|\vartheta\right]\right] - \left(\mathbb{E}\left[\vartheta\right]\right)^2 \\ &= \mathbb{E}\left[\vartheta^2\right] - \left(\mathbb{E}\left[\vartheta\right]\right)^2 \\ &= \text{Var}[\vartheta]. \end{aligned} \tag{1.22}$$

Collecting,

$$\begin{aligned}\widehat{\mathbb{E}}\left[\vartheta|Y=y\right] &= \mathbb{E}\left[\vartheta\right] + \frac{\text{Var}[\vartheta]}{\text{Var}[\vartheta] + \mathbb{E}\left[\vartheta\right]}\left(y - \mathbb{E}\left[\theta\right]\right) \\ &= \frac{\text{Var}[\vartheta]}{\text{Var}[\vartheta] + \mathbb{E}\left[\vartheta\right]} y + \frac{(\mathbb{E}\left[\vartheta\right])^2}{\text{Var}[\vartheta] + \mathbb{E}\left[\vartheta\right]}, \quad y \in \mathbb{R}. \end{aligned} \tag{1.23}$$

5.

5.a. Here $\Theta = \{0, 1, \ldots, M-1\}$. With $\theta = 0, 1, \ldots, M-1$, we have

$$\begin{aligned} f_\theta^{(n)}(y_1, \ldots, y_n) &= \prod_{i=1}^n f_\theta(y_i) \\ &= e^{\sum_{i=1}^n \log f_\theta(y_i)} \\ &= e^{\sum_{k=0}^{M-1} \mathbf{1}[\theta=k]\left(\sum_{i=1}^n \log f_k(y_i)\right)} \\ &= C_n(\theta) q_n(y_1, \ldots, y_n) e^{Q_n(\theta)' K_n(y_1, \ldots, y_n)} \end{aligned} \tag{1.24}$$

where for each θ in Θ, we have set

$$C_n(\theta) = 1 \quad \text{and} \quad Q_n(\theta) = \left(\mathbf{1}\left[\theta = 0\right], \ldots, \mathbf{1}\left[\theta = M-1\right]\right)'$$

while with $y_1, \ldots, y_n$ in $\mathbb{R}$,

$$q_n(y_1, \ldots, y_n) = 1$$

and

$$K_n(y_1, \ldots, y_n) = \left(\sum_{i=1}^n \log f_0(y_i), \ldots, \sum_{i=1}^n \log f_{M-1}(y_i)\right)'.$$

It is plain from (1.24) that the family $\{F_m^{(n)},\ m = 0, \ldots, M-1\}$ is an exponential family.

5.b. There are M natural sufficient statistics $T_0, \ldots, T_{M-1} : \mathbb{R}^n \to \mathbb{R}$ given by

$$T_m(y_1, \ldots, y_n) = \sum_{i=1}^n \log f_m(y_i), \qquad \begin{matrix} y_1, \ldots, y_n \in \mathbb{R} \\ m = 0, \ldots, M-1. \end{matrix}$$

This set of sufficient statistics are marginally interesting as they are equivalent to the statistics $f_0^{(n)}, \ldots, f_{M-1}^{(n)} : \mathbb{R}^n \to \mathbb{R}_+$; however they do reduce dimensionality from n (number of observations) to M (number of hypotheses)!

6.

6.a. With $\eta > 0$, write $\eta = e^{-\tau}$ for some τ in $\mathbb{R}$. The test $d_\eta : \mathbb{R} \to \{0, 1\}$ is then given by

$$d_\eta(y) = 0 \quad \text{iff} \quad f_\lambda(y) < \eta f_0(y).$$

This reduces to

$$d_\eta(y) = 0 \quad \text{iff} \quad \tau + |y| < |y - \lambda|$$

so that

$$C(d_\eta) = \{y \in \mathbb{R} : \ \tau + |y| < |y - \lambda|\} \quad \text{with } \eta = e^{-\tau}.$$

Three separate cases need to be considered. An easy geometric argument shows the following:

(i) If $\tau < -\lambda$, then $C(d_\eta) = \mathbb{R}$.

(ii) If $-\lambda \le \tau < \lambda$, then

$$C(d_\eta) = \left(-\infty, \frac{\lambda - \tau}{2}\right).$$

(iii) If $\lambda \le \tau$, then $C(d_\eta)$ is empty.

6.b. Using the results of Part **a**, we conclude the following: If $\tau < -\lambda$, then

$$P_F(d_\eta) = \mathbb{P}\left[d_\eta(Y) = 1 | H = 0\right] = 0,$$

and

$$P_D(d_\eta) = \mathbb{P}\left[d_\eta(Y) = 1 | H = 1\right] = 0.$$

If $\lambda \le \tau$, then

$$P_F(d_\eta) = \mathbb{P}\left[d_\eta(Y) = 1 | H = 0\right] = 1,$$

and

$$P_D(d_\eta) = \mathbb{P}\left[d_\eta(Y) = 1 | H = 1\right] = 1.$$

If $-\lambda \le \tau < \lambda$, then

$$\begin{aligned} P_F(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 0\right] \\ &= \mathbb{P}\left[Y \ge \frac{\lambda - \tau}{2} \middle| H = 0\right] \\ &= \int_{\frac{\lambda-\tau}{2}}^{\infty} f_0(y) dy \\ &= \int_{\frac{\lambda-\tau}{2}}^{\infty} \frac{1}{2} e^{-|y|} dy \\ &= \frac{1}{2} e^{-\frac{\lambda-\tau}{2}} \end{aligned} \tag{1.25}$$

and

$$\begin{aligned} P_D(d_\eta) &= \mathbb{P}\left[d_\eta(Y) = 1 | H = 1\right] \\ &= \mathbb{P}\left[Y \geq \frac{\lambda - \tau}{2} \Big| H = 1\right] \\ &= \int_{\frac{\lambda-\tau}{2}}^{\infty} f_\lambda(y) dy \\ &= \int_{\frac{\lambda-\tau}{2}}^{\infty} \frac{1}{2} e^{-|y-\lambda|} dy. \end{aligned} \tag{1.26}$$

In computing this last integral, we note that

$$0 < \frac{\lambda - \tau}{2} \leq \lambda \quad \text{if} \quad -\lambda \leq \tau < \lambda.$$

Therefore,

$$\begin{aligned} \int_{\frac{\lambda-\tau}{2}}^{\infty} e^{-|y-\lambda|} dy &= \int_{\frac{\lambda-\tau}{2}}^{\lambda} e^{-|y-\lambda|} dy + \int_{\lambda}^{\infty} e^{-|y-\lambda|} dy \\ &= \int_{\frac{\lambda-\tau}{2}}^{\lambda} e^{-(\lambda-y)} dy + \int_{\lambda}^{\infty} e^{-(y-\lambda)} dy \\ &= e^{-\lambda}\left(e^{\lambda} - e^{\frac{\lambda-\tau}{2}}\right) + 1 \\ &= 2 - e^{-\frac{\lambda+\tau}{2}}, \end{aligned} \tag{1.27}$$

whence

$$P_D(d_\eta) = 1 - \frac{1}{2} e^{-\frac{\lambda+\tau}{2}}.$$

6.c. To compute the ROC curve, first we note that

$$\{P_F(d_\eta),\ \tau < -\lambda\} = \{P_D(d_\eta),\ \tau < -\lambda\} = \{0\}$$

and

$$\{P_F(d_\eta),\ \lambda \leq \tau\} = \{P_D(d_\eta),\ \lambda \leq \tau\} = \{1\},$$

while

$$\{P_F(d_\eta),\ -\lambda \leq \tau < \lambda\} = \left\{\frac{1}{2} e^{-\frac{\lambda-\tau}{2}},\ -\lambda \leq \tau < \lambda\right\} = \left[\frac{e^{-\lambda}}{2}, \frac{1}{2}\right)$$

and

$$\{P_D(d_\eta),\ -\lambda \leq \tau < \lambda\} = \left\{1 - \frac{1}{2} e^{-\frac{\lambda+\tau}{2}},\ -\lambda \leq \tau < \lambda\right\} = \left(\frac{1}{2}, 1 - \frac{1}{2} e^{-\lambda}\right].$$

With P_F in $\left[\frac{e^{-\lambda}}{2}, \frac{1}{2}\right)$ we solve the equation

$$\frac{1}{2} e^{-\frac{\lambda-\tau}{2}} = P_F, \quad -\lambda \leq \tau < \lambda$$

It has a unique solution $\tau(P_F)$ given by

$$\tau(P_F) = \lambda + \log(4P_F^2).$$

Note that

$$\tau(P_F) \in [-\lambda, \lambda)$$

by direct inspection (as expected). Therefore, with $\eta = e^{-\tau(P_F)}$, we find

$$\begin{aligned} P_D(d_\eta) &= 1 - \frac{1}{2} e^{-\frac{\lambda + \tau(P_F)}{2}} \\ &= 1 - \frac{1}{2} e^{-\frac{\lambda + \lambda + \log(4P_F^2)}{2}} \\ &= 1 - \frac{1}{2} e^{-\lambda - \log(2P_F)} \\ &= 1 - \frac{e^{-\lambda}}{4P_F}. \end{aligned} \tag{1.28}$$

From the discussion it follows that the mapping Γ is not defined on the entire interval $[0, 1]$. In fact, we have

$$\Gamma(P_F) = \begin{cases} 0 & \text{if } P_F = 0 \\ 1 - \frac{e^{-\lambda}}{4P_F} & \text{if } P_F \in \left[\frac{e^{-\lambda}}{2}, \frac{1}{2}\right) \\ 1 & \text{if } P_F = 1. \end{cases}$$

As always the ROC curve can be completed through linear interpolation achieved through randomization.
