

## ESTIMATION AND DETECTION THEORY

## HOMEWORK # 8:

Please work out the **ten** (10) problems stated below – HVP refers to the text: H. Vincent Poor, *An Introduction to Signal Detection and Estimation* (Second Edition), Springer Texts in Electrical Engineering Springer, New York (NY), 2010. With this in mind, Exercise **II.2** (HVP) refers to Exercise 2 for Chapter II of HVP. Exercises are located at the end of each chapter.

**Show** work and **explain** reasoning.

1. \_\_\_\_\_  
Solve Exercise **IV.5** (HVP).

2. \_\_\_\_\_  
Assume the family  $\{F_\theta, \theta \in \Theta\}$  to be an exponential family (with respect to some distribution function  $F$  on  $\mathbb{R}^k$ ) with density functions of the form

$$f_\theta(\mathbf{y}) = C(\theta)q(\mathbf{y})e^{Q(\theta)'K(\mathbf{y})} \quad F - \text{a.e.}$$

for every  $\theta$  in  $\Theta$  with Borel mappings  $C : \Theta \rightarrow \mathbb{R}_+$ ,  $Q : \Theta \rightarrow \mathbb{R}^q$ ,  $q : \mathbb{R}^k \rightarrow \mathbb{R}_+$ , and  $K : \mathbb{R}^k \rightarrow \mathbb{R}^q$ .

**2.a** Specialize Conditions **(CR1)**–**(CR5)** in terms (of properties) of the Borel mappings entering the definition of the exponential family. In particular, show that the regularity condition **(CR5)** is equivalent to

$$\left( \frac{\partial}{\partial \theta_i} Q(\theta) \right)' \mathbb{E}_\theta [K(\mathbf{Y})] = -\frac{\partial}{\partial \theta_i} \log C(\theta), \quad \theta_i \in \Theta, \quad i = 1, \dots, p.$$

**2.b** Find an expression for the Fisher information matrix  $M(\theta)$  in terms of the covariance  $\text{Cov}_\theta [K(\mathbf{Y})]$ . **HINT:** Use Part **2.a** together with the fact that

$$\frac{\partial}{\partial \theta_i} \log f_\theta(\mathbf{y}) = \frac{\partial}{\partial \theta_i} \log C(\theta) + \frac{\partial}{\partial \theta_i} Q(\theta)' K(\mathbf{y}), \quad \begin{matrix} i = 1, \dots, p \\ \mathbf{y} \in S \end{matrix}$$

**2.c** What are the conditions that need to hold for an estimator  $g : \mathbb{R}^k \rightarrow \mathbb{R}^p$  to be a regular estimator?

3. \_\_\_\_\_  
Solve Exercise **IV.15** (HVP).

4.

Solve Exercise **IV.21** Part (a) (HVP).

5.

Solve Exercise **IV.22** (HVP).

6.

In the context of the Cramèr-Rao bounds, with arbitrary  $p$ , consider the  $\mathbb{R}^p$ -valued rv  $U(\theta, \mathbf{Y})$  given by

$$U(\theta, \mathbf{Y}) = g(\mathbf{Y}) - \theta - b_\theta(g) - (\mathbf{I}_p + \nabla_\theta b_\theta(g)) M(\theta)^{-1} \nabla_\theta \log f_\theta(\mathbf{Y}), \quad \theta \in \Theta.$$

Show that the rv  $U(\theta, \mathbf{Y})$  has zero mean and that its covariance matrix is given by

$$\begin{aligned} \text{Cov}_\theta[U(\theta, \mathbf{Y})] &= \text{Cov}_\theta[U(\theta, \mathbf{Y})] \\ &= \Sigma_\theta(g) - b_\theta(g)b_\theta(g)' \\ &\quad - (\mathbf{I}_p + \nabla_\theta b_\theta(g)) M(\theta)^{-1} (\mathbf{I}_p + \nabla_\theta b_\theta(g))'. \end{aligned} \quad (1.1)$$

Use this fact to show that the Cramèr-Rao bound is equivalent to the statement that the covariance matrix  $\text{Cov}_\theta[U(\theta, \mathbf{Y})]$  is positive semi-definite. Explore what happens when the bound is achieved.

In Exercises **7** to **9**, a family of distributions  $\{F_\theta, \theta \in \Theta\}$  is given. For each  $n = 1, 2, \dots$ , let  $\{F_\theta^{(n)}, \theta \in \Theta\}$  denote the corresponding families associated with  $n$  i.i.d. samples. In each case, (i) compute the Fisher information matrices  $M^{(n)}(\theta)$  for each  $\theta$  in  $\Theta$ ; (ii) find the efficient estimator for  $\theta$  on the basis of the samples  $Y_1, \dots, Y_n$ ; (iii) find the ML estimator of  $\theta$  on the samples  $Y_1, \dots, Y_n$ ; (iv) Are these estimators unbiased? asymptotically unbiased? consistent? asymptotically normal?

7.

Here  $\Theta = (0, \infty)$  and for each  $\theta > 0$ ,  $F_\theta$  is an exponential distribution with parameter  $\theta$ , namely

$$F_\theta(y) = (1 - e^{-\theta y})^+, \quad y \in \mathbb{R}.$$

8.

Here  $\Theta = \mathbb{R}$  and for each  $\theta > 0$ ,  $F_\theta$  is the shifted Cauchy distribution whose probability density function is given by

$$f_\theta(y) = \frac{1}{\pi(1 + (y - \theta)^2)}, \quad y \in \mathbb{R}.$$

9.

Here  $\Theta = (0, \infty)$  and for each  $\theta > 0$ ,  $F_\theta$  is the Gaussian distribution with zero mean and variance  $\theta$ .

10.

Solve Exercise **IV.23** (HVP).