

ESTIMATION AND DETECTION THEORY

HOMEWORK # 8:

Please work out the **ten** (10) problems stated below – HVP refers to the text: H. Vincent Poor, *An Introduction to Signal Detection and Estimation* (Second Edition), Springer Texts in Electrical Engineering Springer, New York (NY), 2010. With this in mind, Exercise **II.2** (HVP) refers to Exercise 2 for Chapter II of HVP. Exercises are located at the end of each chapter.

Show work and **explain** reasoning.

1. _____
Solve Exercise **IV.5** (HVP).

2. _____
Assume the family $\{F_\theta, \theta \in \Theta\}$ to be an exponential family (with respect to some distribution function F on \mathbb{R}^k) with density functions of the form

$$f_\theta(\mathbf{y}) = C(\theta)q(\mathbf{y})e^{Q(\theta)'K(\mathbf{y})} \quad F - \text{a.e.}$$

for every θ in Θ with Borel mappings $C : \Theta \rightarrow \mathbb{R}_+$, $Q : \Theta \rightarrow \mathbb{R}^q$, $q : \mathbb{R}^k \rightarrow \mathbb{R}_+$, and $K : \mathbb{R}^k \rightarrow \mathbb{R}^q$.

2.a Specialize Conditions **(CR1)–(CR5)** in terms (of properties) of the Borel mappings entering the definition of the exponential family. In particular, show that the regularity condition **(CR5)** is equivalent to

$$\left(\frac{\partial}{\partial \theta_i} Q(\theta) \right)' \mathbb{E}_\theta [K(\mathbf{Y})] = - \frac{\partial}{\partial \theta_i} \log C(\theta), \quad \begin{matrix} \theta_i \in \Theta \\ i = 1, \dots, p \end{matrix}$$

2.b Find an expression for the Fisher information matrix $M(\theta)$ in terms of the covariance $\text{Cov}_\theta [K(\mathbf{Y})]$. **HINT:** Use Part **2.a** together with the fact that

$$\frac{\partial}{\partial \theta_i} \log f_\theta(\mathbf{y}) = \frac{\partial}{\partial \theta_i} \log C(\theta) + \frac{\partial}{\partial \theta_i} Q(\theta)'K(\mathbf{y}), \quad \begin{matrix} i = 1, \dots, p \\ \mathbf{y} \in S \end{matrix}$$

2.c What are the conditions that need to hold for an estimator $g : \mathbb{R}^k \rightarrow \mathbb{R}^p$ to be a regular estimator?

3. _____
Solve Exercise **IV.15** (HVP).

4. _____

Solve Exercise **IV.21** Part (a) (HVP).

5. _____

Solve Exercise **IV.22** (HVP).

6. _____

In the context of the Cramèr-Rao bounds, with arbitrary p , consider the \mathbb{R}^p -valued rv $U(\theta, \mathbf{Y})$ given by

$$U(\theta, \mathbf{Y}) = g(\mathbf{Y}) - \theta - b_\theta(g) - (\mathbf{I}_p + \nabla_\theta b_\theta(g)) M(\theta)^{-1} \nabla_\theta \log f_\theta(\mathbf{Y}), \quad \theta \in \Theta.$$

Show that the rv $U(\theta, \mathbf{Y})$ has zero mean and that its covariance matrix is given by

$$\begin{aligned} \text{Cov}_\theta[U(\theta, \mathbf{Y})] &= \text{Cov}_\theta[U(\theta, \mathbf{Y})] \\ &= \Sigma_\theta(g) - b_\theta(g) b_\theta(g)' \\ &\quad - (\mathbf{I}_p + \nabla_\theta b_\theta(g)) M(\theta)^{-1} (\mathbf{I}_p + \nabla_\theta b_\theta(g))'. \end{aligned} \quad (1.1)$$

Use this fact to show that the Cramèr-Rao bound is equivalent to the statement that the covariance matrix $\text{Cov}_\theta[U(\theta, \mathbf{Y})]$ is positive semi-definite. Explore what happens when the bound is achieved.

In Exercises **7** to **9**, a family of distributions $\{F_\theta, \theta \in \Theta\}$ is given. For each $n = 1, 2, \dots$, let $\{F_\theta^{(n)}, \theta \in \Theta\}$ denote the corresponding families associated with n i.i.d. samples. In each case, (i) compute the Fisher information matrices $M^{(n)}(\theta)$ for each θ in Θ ; (ii) find the efficient estimator for θ on the basis of the samples Y_1, \dots, Y_n ; (iii) find the ML estimator of θ on the samples Y_1, \dots, Y_n ; (iv) Are these estimators unbiased? asymptotically unbiased? consistent? asymptotically normal?

7. _____

Here $\Theta = (0, \infty)$ and for each $\theta > 0$, F_θ is an exponential distribution with parameter θ , namely

$$F_\theta(y) = (1 - e^{-\theta y})^+, \quad y \in \mathbb{R}.$$

8. _____

Here $\Theta = \mathbb{R}$ and for each $\theta > 0$, F_θ is the shifted Cauchy distribution whose probability density function is given by

$$f_\theta(y) = \frac{1}{\pi(1 + (y - \theta)^2)}, \quad y \in \mathbb{R}.$$

9. _____

Here $\Theta = (0, \infty)$ and for each $\theta > 0$, F_θ is the Gaussian distribution with zero mean and variance θ .

10. _____

Solve Exercise **IV.23** (HVP).
