ɔy

$P'_D(\tilde{\delta};\theta_0)$. (II.E.28)

nate maximum power with size
test that maximizes $P'_D(\tilde{\delta};\theta_0)$
α, is called an α-level *locally*
lly optimum test.
sts we note that, assuming that
write

')}

$)p_\theta(y)\mu(dy)$. (II.E.29)

$\theta \in \Lambda_1\}$ that we can interchange
(II.E.29), we have

$)_\theta(y)\,|_{\theta=\theta_0}\mu(dy)$. (II.E.30)

licates that the α-level LMP design
nan-Pearson design problem, if we
g this analogy, it is straightforward
vel LMP test for (II.E.26) is given

$$\theta_0 \begin{matrix} > \\ = \\ < \end{matrix} \eta p_{\theta_0}(y), \quad \text{(II.E.31)}$$

$_{l_0}) = \alpha$. Details of this development
on (1968). LMP tests are discussed

y of the above-mentioned optimality
composite problems in which θ is the
ised on comparing the quantity

$$\frac{p_\theta(y)}{p_\theta(y)} \quad \text{(II.E.32)}$$

es known as the *generalized likelihood-*
est, and further motivation for tests of

II.F Exercises

1. Find the minimum Bayes risk for the binary channel of Example II.B.1.

2. Suppose Y is a random variable that, under hypothesis H_0, has pdf

$$p_0(y) = \begin{cases} (2/3)(y+1), & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

and, under hypothesis H_1, has pdf

$$p_1(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the Bayes rule and minimum Bayes risk for testing H_0 versus H_1 with uniform costs and equal priors.

   (b) Find the minimax rule and minimax risk for uniform costs.

   (c) Find the Neyman-Pearson rule and the corresponding detection probability for false-alarm probability $\alpha \in (0,1)$.

3. Repeat Exercise 2 for the situation in which p_j is given instead by

$$p_j(y) = \frac{(j+1)}{2} e^{-(j+1)|y|}, \; y \in \mathbb{R}, j = 0, 1.$$

For parts (a) and (b) assume costs

$$C_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i = 1 \text{ and } j = 0 \\ 3/4, & \text{if } i = 0 \text{ and } j = 1, \end{cases}$$

and for part (a) assume priors $\pi_0 = 1/4$ and $\pi_1 = 3/4$.

4. Repeat Exercise 2 for the situation in which p_0 and p_1 are given instead by

$$p_0(y) = \begin{cases} e^{-y}, & y \ge 0 \\ 0, & y < 0 \end{cases}$$

and

$$p_1(y) = \begin{cases} \sqrt{2/\pi}\, e^{-y^2/2}, & y \ge 0 \\ 0, & y < 0. \end{cases}$$

For part (a) consider arbitrary priors.

5. Repeat Exercise 2 for the hypothesis pair

$$H_0 : Y \text{ has density } p_0(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, y \in \mathbb{R}$$

versus

$$H_1 : Y \text{ has density } p_1(y) = \begin{cases} 1/5, & \text{if } y \in [0,5] \\ 0, & \text{if } y \notin [0,5]. \end{cases}$$

For part (a) assume priors $\pi_0 = 3/4$ and $\pi_1 = 1/4$.

6. Repeat Exercise 2 for the hypothesis pair

$$H_0 : Y = N - s$$

versus

$$H_1 : Y = N + s$$

where $s > 0$ is a fixed real number and N is a continuous random variable with density

$$p_N(n) = \frac{1}{\pi(1+n^2)}, \quad n \in \mathbb{R}.$$

7. (a) Consider the hypothesis pair

$$H_0 : Y = N$$

versus

$$H_1 : Y = N + S$$

where N and S are independent random variables each having pdf

$$p(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Find the likelihood ratio between H_0 and H_1.

(b) Find the threshold and detection probability for α-level Neyman-Pearson testing in (a).

(c) Consider the hypothesis pair

$$H_0 : Y_k = N_k, \quad k = 1, \ldots, n$$

versus

$$H_1 : Y_k = N_k + S, \quad k = 1, \ldots, n$$

where $n > 1$ and $N_1, \ldots, N_n$, and S are independent random variables each having the pdf given in (a). Find the likelihood ratio.

(d) Find the threshold for α-level Neyman-Pearson testing in (c).

8. Show that the minimum-Bayes-risk function V (defined in Section II.C) is concave and continuous in [0, 1]. [After showing that V is concave you may use the fact that any concave function on [0, 1] is continuous on (0, 1).]

9. Suppose we have a real observation Y and binary hypotheses described by the following pair of pdf's:

$$p_0(y) = \begin{cases} (1 - |y|), & \text{if } |y| \le 1 \\ 0, & \text{if } |y| > 1 \end{cases}$$

and

$$p_1(y) = \begin{cases} (2 - |y|)/4, & \text{if } |y| \le 2 \\ 0, & \text{if } |y| > 2 \end{cases}$$

(a) Assume that the costs are given by

$$\begin{aligned} C_{01} &= 2C_{10} > 0 \\ C_{00} &= C_{11} = 0. \end{aligned}$$

Find the minimax test of H_0 versus H_1 and the corresponding minimax risk.

(b) Find the Neyman-Pearson test of H_0 versus H_1 with false-alarm probability α. Find the corresponding power of the test.

10. Suppose we observe a random variable Y given by

$$Y = N + \theta\lambda$$

where θ is either 0 or $1, \lambda$ is a fixed number between 0 and 2, and where N is a random variable that has a uniform density on the interval $(-1, 1)$. We wish to decide between the hypotheses

$$H_0 : \theta = 0$$

versus

$$H_1 : \theta = 1.$$

(a) Find the Neyman-Pearson decision rule for false-alarm probability ranging from 0 to 1.

(b) Find the power of the Neyman-Pearson decision rule as a function of the false-alarm probability and the parameter λ. Sketch the receiver operating characteristics.

11. Consider the simple hypothesis testing problem for the real-valued observation Y:

$$H_0 : p_0(y) = \exp(-y^2/2)/\sqrt{2\pi}, \qquad y \in \mathbb{R}$$

$$H_1 : p_1(y) = \exp(-(y-1)^2/2)/\sqrt{2\pi}, \quad y \in \mathbb{R}.$$

Suppose the cost assignment is given by $C_{00} = C_{11} = 0, C_{10} = 1$, and $C_{01} = N$. Investigate the behavior of the Bayes rule and risk for equally likely hypotheses and the minimax rule and risk when N is very large.

12. Consider a simple binary hypothesis testing problem. For a decision rule δ, denote the false-alarm and miss probabilities by $P_F(\delta)$ and $P_M(\delta)$, respectively. Consider the performance measure:

$$\rho(\delta) \triangleq [P_F(\delta)]^2 + [P_M(\delta)]^2 \,;$$

and let δ_o denote a decision rule minimizing $\rho(\delta)$ over all randomized decision rules δ.

(a) Show that δ_o must be a likelihood-ratio test.

(b) For $\pi_0 \in [0,1]$, define the function V by

$$V(\pi_0) = \min_{\delta} [\pi_0 P_F + (1-\pi_0) P_M] \,.$$

Suppose that $V(\pi_0)$ achieves its maximum on [0,1] at the point $\pi_0 = 1/2$. Show that δ_o is a Bayes rule for prior $\pi_0 = 1/2$. [Hint: Note that we can write $2\rho(\delta) = [P_F(\delta) + P_M(\delta)]^2 + [P_F(\delta) - P_M(\delta)]^2$.]

13. Consider the following Bayes decision problem: The conditional density of the real observation Y given the real parameter $\Theta = \theta$ is given by

$$p_\theta(y) = \begin{cases} \theta e^{-\theta y}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

Θ is random variable with density

$$w(\theta) = \begin{cases} \alpha e^{-\alpha \theta}, & \theta \geq 0 \\ 0, & \theta < 0. \end{cases}$$

where $\alpha > 0$. Find the Bayes rule and minimum Bayes risk for the hypotheses

$$H_0 : \Theta \in (0, \beta) \triangleq \Lambda_0$$

versus

$$H_1 : \Theta \in [\beta, \infty) \triangleq \Lambda_1$$

where $\beta > 0$ is fixed. Assume the cost structure

$$C[i,\theta] = \begin{cases} 1, & \text{if } \theta \notin \Lambda_i \\ 0, & \text{if } \theta \in \Lambda_i. \end{cases}$$

14. Repeat Exercise 13 for the case in which Y consists of n independent (conditioned on Θ) and identically distributed observations $Y = Y_1, \ldots, Y_n$ each with the conditional density given in 13. You need not find the Bayes risk in closed form.

15. Consider the composite hypothesis testing problem:

$$H_0 : Y \text{ has density } p_0(y) = \tfrac{1}{2}e^{-|y|}, \quad y \in \mathbb{R}$$

versus

$$H_1 : Y \text{ has density } p_\theta(y) = \tfrac{1}{2}e^{-|y-\theta|}, y \in \mathbb{R}, \theta > 0.$$

   (a) Describe the locally most powerful α-level test and derive its power function.

   (b) Does a uniformly most powerful test exist? If so, find it and derive its power function. If not, find the generalized likelihood ratio test for H_0 versus H_1.

16. In Section B, we formulated and solved the binary Bayesian hypothesis-testing problem. Generalize this formulation and solution to M hypotheses for $M > 2$.

17. Formulate the M-ary minimax hypothesis-testing problem. Show that a Bayes equalizer rule (if one exists) is minimax.

18. How would you formulate a criterion analogous to the Neyman-Pearson criterion for M hypotheses? Conjecture a solution.

19. Consider the following pair of hypotheses concerning a sequence $Y_1, Y_2, \ldots, Y_n$ of independent random variables

$$H_0 : Y_k \sim \mathcal{N}(\mu_0, \sigma_0^2), \quad k = 1, 2, \ldots, n$$

versus

$$H_1 : Y_k \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad k = 1, 2, \ldots, n$$

where μ_0, μ_1, σ_0^2, and σ_1^2 are known constants.

   (a) Show that the likelihood ratio can be expressed as a function of the parameters μ_0, μ_1, σ_0^2, and σ_1^2, and the quantities $\sum_{k=1}^n Y_k^2$ and $\sum_{k=1}^n Y_k$.

   (b) Describe the Neyman-Pearson test for the two cases ($\mu_0 = \mu_1, \sigma_1^2 > \sigma_0^2$) and ($\sigma_0^2 = \sigma_1^2, \mu_1 > \mu_0$).

   (c) Find the threshold and ROC's for the case $\mu_0 = \mu_1, \sigma_1^2 > \sigma_0^2$ with $n = 1$.

20. Consider the hypotheses of Exercise 19 with $\mu \triangleq \mu_1 > \mu_0 = 0$ and $\sigma^2 \triangleq \sigma_0^2 = \sigma_1^2 > 0$. Does there exist a uniformly most powerful test

of these hypotheses under the assumption that μ is known and σ^2 is not? If so, find it and show that it is UMP. If not, show why and find the generalized likelihood ratio test.

21. Suppose $Y_1, Y_2, \ldots, Y_n$ is a sequence of random observations, each taking the values 0 and 1 with probabilities 1/2. Consider the following two hypotheses concerning $Y_1, Y_2, \ldots, Y_n$:

$$H_0 : Y_1, Y_2, \ldots, Y_n \text{ are independent}$$

versus

$$H_1 : p_1(y_k | y_1, y_2, \ldots, y_{k-1}) = \begin{cases} 3/4 & \text{if } y_k = y_{k-1} \\ 1/4 & \text{if } y_k \neq y_{k-1} \end{cases}, \quad k = 2, 3, \ldots, n,$$

where $p_1(y_k|y_1, y_2, \ldots, y_{k-1})$ denotes the conditional probability that $Y_k = y_k$ given that $Y_1 = y_1, Y_2 = y_2, \ldots, Y_{k-1} = y_{k-1}$. Find the Bayes decision rule for testing H_0 versus H_1 under the assumption of uniform costs and equal priors.