ENEE 621 Spring 2016 Homework 1 Solutions

1. The likelihood ratio can be written as

$$L(1) = \frac{1-\lambda_1}{\lambda_0}, \quad L(0) = \frac{\lambda_1}{1-\lambda_0}$$

From II.B.9, the threshold is

$$\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$$

The Bayes risk can be written as

$$r(\delta) = \sum_{j=0}^{1} \pi_j C_{0j} + \sum_{j=0}^{1} \pi_j (C_{1j} - C_{0j}) P_j(\Gamma_1)$$

from II.B.5. We only need to calculate $P_j(\Gamma_1)$

1° if $L(1) \geq \tau$ and $L(0) \geq \tau$,
then $\Gamma_1 = \{0, 1\}$, $P_j(\Gamma_1) = 1$ for $j = 0, 1$

2° if $L(1) \geq \tau$ and $L(0) < \tau$
then $\Gamma_1 = \{1\}$, $P_0(\Gamma_1) = \lambda_0$, $P_1(\Gamma_1) = 1-\lambda_1$

3° if $L(1) < \tau$ and $L(0) < \tau$
then $\Gamma_1 = \phi$, $P_0(\Gamma_1) = 0$, $P_1(\Gamma_1) = 0$

4° if $L(1) < \tau$ and $L(0) \geq \tau$
then $\Gamma_1 = \{0\}$, $P_0(\Gamma_1) = 1-\lambda_0$, $P_1(\Gamma_1) = \lambda_1$

2. The likelihood ratio can be written as

$$L(y) = \frac{3}{2(y+1)}, \quad 0 \le y \le 1$$

Under equal prior and uniform cost, $\tau = 1$

Therefore,

$$\Gamma_1 = \{ y \in [0,1] \mid L(y) \ge 1 \}$$

$$= \{ y \mid y \in [0, \tfrac{1}{2}] \}$$

The Bayes rule is

$$\delta_B(y) = \begin{cases} 1 & \text{if } 0 \le y \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < y \le 1 \end{cases}$$

The Bayes risk is

$$r(\delta_B) = \frac{1}{2} \int_0^{\frac{1}{2}} \frac{2}{3}(y+1)\,dy + \frac{1}{2} \int_{\frac{1}{2}}^{1} dy$$

$$= \frac{11}{24}$$

3. We have

$$p_0(y) = \begin{cases} \frac{1}{2}e^{-y} & y \geq 0 \\ \frac{1}{2}e^{y} & y < 0 \end{cases}$$

$$p_1(y) = \begin{cases} e^{-2y} & y \geq 0 \\ e^{2y} & y < 0 \end{cases}$$

The threshold is $\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})} = \frac{4}{9}$

The likelihood ratio is

$$L(y) = \frac{p_1(y)}{p_0(y)} = \begin{cases} 2e^{-y} & y \geq 0 \\ 2e^{y} & y < 0 \end{cases}$$

Therefore, $\Gamma_1(y) = \{y \mid L(y) \geq \tau\}$

$$= \{y \mid \ln\frac{2}{9} \leq y \leq \ln\frac{9}{2}\}$$

The Bayes rule is

$$\delta_B(y) = \begin{cases} 1 & \text{if } \ln\frac{2}{9} \leq y \leq \ln\frac{9}{2} \\ 0 & \text{otherwise} \end{cases}$$

The Bayes risk is

$$r(\delta_B) = \frac{1}{4} \times 1 \times \int_{\ln\frac{2}{9}}^{\ln\frac{9}{2}} \frac{1}{2} e^{-|y|} dy$$

$$+ \frac{3}{4} \times \frac{3}{4} \times \left(\int_{-\infty}^{\ln\frac{2}{9}} e^{2y} dy + \int_{\ln\frac{9}{2}}^{\infty} e^{-2y} dy \right)$$

4. Let $x_\lambda = (1-\lambda)x_0 + \lambda x_1$, then we have

$$f_\alpha(x_\lambda) \geq (1-\lambda) f_\alpha(x_0) + \lambda f_\alpha(x_1), \ \forall \alpha \in A$$

Then,

$$\inf(f_\alpha(x_\lambda) : \alpha \in A) \geq \inf((1-\lambda)f_\alpha(x_0) + \lambda f_\alpha(x_1) : \alpha \in A)$$

Since the LHS is $g(x_\lambda)$,

$$\Rightarrow \quad g(x_\lambda) \geq \inf((1-\lambda)f_\alpha(x_0) + \lambda f_\alpha(x_1) : \alpha \in A)$$

$$\geq \inf((1-\lambda)f_\alpha(x_0) : \alpha \in A) + \inf(\lambda f_\alpha(x_1) : \alpha \in A)$$

$$= (1-\lambda)g(x_0) + \lambda g(x_1)$$

5. WLOG, let $a<u<v<w<b$.

By definition (1.1)

$$g(v)-\frac{g(v)-g(u)}{v-u}(v-x)\leq g(x)\leq g(v)-\frac{g(w)-g(v)}{w-v}(v-x)$$

for $x\in[u,v]$

Therefore, we have

$$\lim_{x\to v^-} g(x)=g(v).$$

Similarly,

$$g(v)+\frac{g(v)-g(u)}{v-u}(x-v)\geq g(x)\geq g(v)+\frac{g(w)-g(v)}{w-v}(x-v)$$

for $x\in[v,w]$,

and $\lim_{x\to v^+} g(x)=g(v)$

6. It's not necessarily concave, a counterexample is below.

Let I be $[0, +\infty)$

$$g(x) = \begin{cases} -x^2 & \text{if } x > 0 \\ -1 & x = 0 \end{cases}$$

$\lim_{x \to 0^+} g(x) = 0$ and $g(0) = -1$, which means it's not continuous.

It's easy to verify that $g(x)$ is concave using definition (1.1)

7. The likelihood ratio is

$$L(y) = \sqrt{\frac{2}{\pi}} e^{y - \frac{y^2}{2}} = \sqrt{\frac{2e}{\pi}} e^{-\frac{(y-1)^2}{2}}, \quad y \geq 0$$

We let $\tau' = -\sqrt{\frac{\pi}{2e}} \log\left(\frac{\pi_0}{1-\pi_0}\right)$,

$$\Gamma_1 = \{ y \geq 0 \mid (y-1)^2 \leq \tau' \}$$

there are three cases :

$$\Gamma_1 = \begin{cases} \phi & \text{if } \tau' < 0 \\ [1-\sqrt{\tau'}, 1+\sqrt{\tau'}] & \text{if } 0 \leq \tau' \leq 1 \\ [0, 1+\sqrt{\tau'}] & \text{if } \tau' > 1 \end{cases}$$

The minimum Bayes risk can be calculated for the three cases :

1° $\tau' < 0$, $\quad r(\delta_B) = 1 - \pi_0$

2° $0 \leq \tau' \leq 1$, $\quad r(\delta_B) = \pi_0 \int_{1-\sqrt{\tau'}}^{1+\sqrt{\tau'}} e^{-y}\, dy$

$$+ (1-\pi_0)\sqrt{\frac{2}{\pi}} \left[\int_0^{1-\sqrt{\tau'}} e^{-\frac{y^2}{2}} dy + \int_{1+\sqrt{\tau'}}^{\infty} e^{-\frac{y^2}{2}} dy \right]$$

3° $\tau' > 1$, $\quad r(\delta_B) = \pi_0 \int_0^{1+\sqrt{\tau'}} e^{-y} dy + (1-\pi_0)\sqrt{\frac{2}{\pi}} \int_{1+\sqrt{\tau'}}^{\infty} e^{-\frac{y^2}{2}} dy$

8. Since $\{0,1\} \cap \{y \mid y \in (0,1)\} = \phi$,

the test that minimizes the probability of error can be written as

$$\delta(y) = \begin{cases} 1 & \text{if } y \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

The probability of error is 0

9. It can be formalated as a binary hypothesis testing problem by creating a Borel mapping $d: R^k \longrightarrow \{0,1\}$

$$F_0(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln y - m)^2}{2\sigma^2}} & \text{if } y > 0 \end{cases}$$

$$F_1(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\sqrt{y} - m)^2}{2\sigma^2}} + \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(-\sqrt{y} - m)^2}{2\sigma^2}} & \text{if } y \ge 0 \end{cases}$$

10. The likelihood ratio can be written as

$$L(y) = \frac{F_1(y)}{F_0(y)} = \frac{\frac{y}{\sigma_1^2} e^{-\frac{y^2}{2\sigma_1^2}}}{\frac{y}{\sigma_2^2} e^{-\frac{y^2}{2\sigma_0^2}}}$$

$$= \frac{\sigma_0^2}{\sigma_1^2} e^{\left(\frac{y^2}{2\sigma_0^2} - \frac{y^2}{2\sigma_1^2}\right)} \quad \text{for } y \geq 0$$

the threshold is

$$\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})}$$

The likelihood ratio test is

$$\delta(y) = \begin{cases} 1 & \text{if } L(y) \geq \tau \\ 0 & \text{if } L(y) < \tau \end{cases}$$