ENEE 621 Spring 2016 Homework 4 Solutions

1. a) WOLG, we assume that $a>0$

The likelihood ratio can be written as

$$L(y)=\frac{p_1(y)}{p_0(y)}=e^{y-\frac{a^2}{2}}$$

the threshold for the prior p is

$$\tau=\frac{p}{1-p}$$

Let y_p denote the solution of $e^{y-\frac{a^2}{2}}=\frac{p}{1-p}$,

$$V(p)=p\,R_0(\delta_p)+(1-p)R_1(\delta_p)$$

$$=p\,\phi(y_p)+(1-p)\,\phi(a-y_p)$$

where $\phi(\cdot)$ is the Q-function.

b) Since $\phi(y_p)$ and $\phi(a-y_p)$ are differentiable,

$V:[0,1]\to[0,1]$ is a differentiable function

By (1.59) in the lecture notes, the mapping

$p\to J_p(d)$ is affine since $P_F(d)$ and $P_M(d)$

do not depend on p. Therefore, the mapping

V being infimum of the family $\{J_p(d), d\in D\}$

is also concave. For more information, refer to Lemma (1.11.1) in the lecture notes.

c) Under p_m, the decision rule satisfies

$$P_0(\delta_{p_m}) = P_1(\delta_{p_m})$$

$$\phi(y_{p_m}) = \phi(a - y_{p_m})$$

$y_{p_m} = \frac{a}{2}$, and p_m satisfies

$$\frac{p_m}{1-p_m} = e^{\frac{a-a^2}{2}}$$

2. b) The likelihood ratio can be written as

$$L(y) = \frac{3}{2(y+1)} \quad , \quad 0 \le y \le 1$$

With uniform costs, we can have

$$\Gamma_1 = \left\{ y \in [0,1] \,\middle|\, L(y) \ge \frac{\pi_0}{1-\pi_0} \right\} = [0, \tau']$$

where

$$\tau' = \begin{cases} 1 & \text{if } 0 \le \pi_0 \le \frac{3}{7} \\ \frac{1}{2}\left(\frac{3}{\pi_0} - 5\right) & \text{if } \frac{3}{7} < \pi_0 < \frac{3}{5} \\ 0 & \text{if } \frac{3}{5} \le \pi_0 \le 1 \end{cases}$$

thus, the conditional risks are

$$R_0(\delta_{\pi_0}) = \int_0^{\tau'} \frac{2}{3}(y+1)\, dy = \begin{cases} 1 & \text{if } \pi_0 \in [0, \frac{3}{7}] \\ \frac{2}{3}\tau'(\frac{\tau'}{2}+1) & \text{if } \pi_0 \in (\frac{3}{7}, \frac{3}{5}) \\ 0 & \text{if } \pi_0 \in [\frac{3}{5}, 1] \end{cases}$$

$$R_1(\delta_{\pi_0}) = \int_{\tau'}^{1} dy = \begin{cases} 0 & \text{if } \pi_0 \in [0, \frac{3}{7}] \\ 1-\tau' & \text{if } \pi_0 \in (\frac{3}{7}, \frac{3}{5}) \\ 1 & \text{if } \pi_0 \in [\frac{3}{5}, 1] \end{cases}$$

The minimax threshold τ_L' satisfies

$$\frac{2}{3}\tau_L'\left(\frac{\tau_L'}{2}+1\right) = 1 - \tau_L'$$

The minimum risk is $V(\pi_L) = \pi_L R_0(\delta_{\pi_L}) + (1-\pi_L) R_1(\delta_{\pi_L})$

$= 1 - \pi_L^2$

c) Since $L(y)$ is monotone decreasing in y,

the test can be written as

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } y < \eta' \\ 0 & \text{if } y \geq \eta' \end{cases}$$

The false alarm probability

$$P_F = p_0(Y < \eta') = \begin{cases} 0 & \text{if } \eta' \leq 0 \\ \frac{2}{3}\eta'(\frac{\eta'}{2}+1) & \text{if } 0 < \eta' < 1 \\ 1 & \text{if } \eta' \geq 1 \end{cases}$$

and we can have

$$\frac{2}{3}\eta'(\frac{\eta'}{2}+1) = \alpha \quad , \eta' = \sqrt{1+3\alpha} - 1$$

the test is $\delta_{NP}(y) = \begin{cases} 1 & \text{if } y \leq \eta' \\ 0 & \text{if } y > \eta' \end{cases}$

$$P_D = \int_0^{\eta'} dy = \eta'$$

3. The likelihood ratio can be written as

$$L(y)=\begin{cases} 2e^{-y} & y\geq 0 \\ 2e^{y} & y<0 \end{cases}$$

the threshold is $\tau=\frac{\pi_0\,(C_{10}-C_{00})}{\pi_1\,(C_{01}-C_{11})}=\frac{4\pi_0}{3(1-\pi_0)}$.

We can have $\Gamma_1(y)=\begin{cases}\{y\,|\,-\tau'\leq y\leq \tau'\} & \text{if } \pi_0\leq\frac{3}{5} \\ \phi & \text{otherwise}\end{cases}$

where $\tau'=-\ln\frac{2\pi_0}{3(1-\pi_0)}$

Therefore, $R_0(\delta_{\pi_0})=\begin{cases}\int_{-\tau'}^{\tau'}\frac{1}{2}e^{-|y|}dy & \text{if } \pi_0\leq\frac{3}{5} \\ 0 & \text{otherwise}\end{cases}$

$$R_1(\delta_{\pi_0})=\begin{cases}\frac{3}{4}\left(\int_{-\infty}^{-\tau'}e^{2y}dy+\int_{\tau'}^{\infty}e^{-2y}dy\right) & \text{if } \pi_0\leq\frac{3}{5} \\ \frac{3}{4} & \text{otherwise}\end{cases}$$

The minimax threshold τ_L' satisfies

$$\int_{-\tau_L'}^{\tau_L'}\frac{1}{2}e^{-|y|}dy=\frac{3}{4}\left(\int_{-\infty}^{-\tau_L'}e^{2y}dy+\int_{\tau_L'}^{\infty}e^{-2y}dy\right)$$

The minimum risk $V(\pi_L)=\pi_L R_0(\delta_{\pi_L})+(1-\pi_L)R_1(\delta_{\pi_L})$

$$=\int_{-\tau_L'}^{\tau_L'}\frac{1}{2}e^{-|y|}dy$$

c) The Neyman-Pearson decision rule can be written as

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } -\eta' \leq y \leq \eta' \\ 0 & \text{otherwise} \end{cases}$$

$$P_F = \int_{-\eta'}^{\eta'} \frac{1}{2} e^{-|y|} \, dy = 1 - e^{-\eta'}$$

$$P_F = \alpha \Rightarrow \quad 1 - e^{-\eta'} = \alpha$$

$$\eta' = -\ln(1-\alpha)$$

the detection probability is

$$P_D = \int_{-\infty}^{-\eta'} e^{2y} \, dy + \int_{\eta'}^{\infty} e^{-2y} \, dy$$

4. The likelihood ratio is

$$L(y) = \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}+y} \quad , y \geq 0$$

the threshold is $\tau = \frac{\pi_0}{1-\pi_0}$

If $\pi_0 > \frac{1}{1+\sqrt{\frac{\pi}{2e}}}$, we will always decide H_0 is true,

then $V(\pi_0) = 1-\pi_0$

If $\pi_0 \leq \frac{1}{1+\sqrt{\frac{\pi}{2e}}}$, we will decide H_1 is true if

$$A' = 1-\sqrt{1-2\log\sqrt{\frac{\pi}{2}} \cdot \frac{\pi_0}{1-\pi_0}} \leq y \leq 1+\sqrt{1-2\log\sqrt{\frac{\pi}{2}} \cdot \frac{\pi_0}{1-\pi_0}} = B'.$$

Therefore, $P_0(\delta_{\pi_0}) = \int_{A'}^{B'} e^{-y} dy = e^{-A'} - e^{-B'}$

$$P_1(\delta_{\pi_0}) = \int_0^{A'} \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy + \int_{B'}^{\infty} \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy$$

The least favorable prior π_L satisfies

$$P_0(\delta_{\pi_L}) = P_1(\delta_{\pi_L}),$$

and the minimum risk is equal to $e^{-A'} - e^{-B'}$

when $\pi_0 = \pi_L$

c) Recall from problem 7 of Homework 1,

$$\Gamma_1 = \left\{ y \geq 0 \,\middle|\, (y-1)^2 \leq \eta' \right\}$$

where $\eta' = -\sqrt{\frac{\pi}{2e}} \log(\eta)$

The false alarm probability is

$$P_F(\delta_{NP}) = 0 \quad \text{if } \eta' < 0$$

$$P_F(\delta_{NP}) = \int_{1-\sqrt{\eta'}}^{1+\sqrt{\eta'}} e^{-y} dy = \frac{2}{e} \sinh(\sqrt{\eta'}) \quad \text{if } 0 < \eta' \leq 1$$

$$P_F(\delta_{NP}) = \int_0^{1+\sqrt{\eta'}} e^{-y} dy = 1 - e^{-1-\sqrt{\eta'}}, \quad \eta' > 1$$

Therefore,

$$\eta' = \begin{cases} \left(\sinh^{-1}\left(\frac{\alpha e}{2}\right)\right)^2 & \text{if } 0 < \alpha \leq 1 - e^{-2} \\ (1 + \log(1-\alpha))^2 & \text{if } 1 - e^{-2} < \alpha < 1 \end{cases}$$

$$P_D(\delta_{NP}) = 2(\phi(1+\sqrt{\eta'}) - \phi(1-\sqrt{\eta'})) \quad \text{if } 0 < \alpha \leq 1 - e^{-2}$$

$$P_D(\delta_{NP}) = 2\left(\phi(1+\sqrt{\eta'}) - \frac{1}{2}\right) \quad \text{if } 1 - e^{-2} < \alpha \leq 1$$

5. b) Recall from problem 1 in homework 2, the likelihood ratio is

$$L(y) = \frac{\sqrt{2\pi}}{5} e^{\frac{y^2}{2}}, \quad y \in [0,5]$$

Obviously we have to decide H_0 if $y > 5$.

Therefore, $\Gamma_1 = [\eta', 5]$

the conditional costs are

$$R_0(\delta) = \phi(\eta') - \phi(5)$$

$$R_1 = \frac{\eta'}{5}$$

For the minimax test, η' satisfies

$$\phi(\eta') - \phi(5) = \frac{\eta'}{5}$$

and π_L satisfies

$$L(\eta') = \frac{\pi_L}{1-\pi_L}$$

The minimum risk is $V(\pi_L) = \pi_L R_0(\delta_{\pi_L}) + (1-\pi_L) R_1(\delta_{\pi_L})$

$$= \frac{\eta'}{5}$$

c) The false alarm probability is

$$P_F = \phi(\eta') - \phi(5)$$

For the NP test, if $\alpha \geq \phi(0) - \phi(5)$

$$= \frac{1}{2} - \phi(5).$$

then $\Gamma_1 = [0, 5]$, $P_D = 1$

If $0 < \alpha < \frac{1}{2} - \phi(5)$, the $\Gamma_1 = [\eta', 5]$

where $\phi(\eta') - \phi(5) = \alpha$

$$P_D = 1 - \frac{\eta'}{5}$$

6. b) Recall from problem 3 in homework 2,

$L(y) = y \ , \ y \geq 0$

Therefore, $\Gamma_1 = [0, \eta']$

$$P_F = \int_{\eta'}^{\infty} e^{-y} \, dy$$

$$= e^{-\eta'}$$

The α-level Neyman-Pearson test is

$\Gamma_1 = [0, \eta']$ where $e^{-\eta'} = \alpha$

$\eta' = -\log \alpha$

$$P_D = \int_{\eta'}^{\infty} p_1(y) \, dy = \int_{\eta'}^{\infty} y e^{-y} \, dy = \alpha(1 - \log \alpha)$$

c) Here the densities under the two hypotheses are

$$p_0(\underline{y}) = \prod_{k=1}^{n} p(y_k) = \prod_{k=1}^{n} e^{-y_k}$$

since $N_1 \cdots N_n$ are independent.

Under H_1, $y_1 \cdots y_n$ are NOT independent due to the common S.

$$p_1(\underline{y}) = \int_{-\infty}^{\infty} \left[\prod_{k=1}^{n} p(y_k - s)\right] p(s)\, ds$$

$$= \int_0^{\underline{y}} \left[\prod_{k=1}^{n} p(y_k - s)\right] e^{-s}\, ds$$

$$= \frac{p_0(\underline{y})}{n-1} \left[e^{(n-1)\underline{y}} - 1\right]$$

where $\underline{y} = \min\{y_1 \cdots y_n\}$

Thus, $L(\underline{y}) = \dfrac{p_1(\underline{y})}{p_0(\underline{y})} = \dfrac{1}{n-1}\left[e^{(n-1)\underline{y}} - 1\right]$

d) $P_F = P_0(L(y) > \tau')$

$= P_0(\underline{y} > \eta')$

$= P_0(y_1 > \eta', y_2 > \eta' \cdots y_n > \eta')$

$= e^{-n\eta'}$

due to that $N_1 \cdots N_n$ are independent.

$$\eta' = \frac{\log((n-1)\tau' + 1)}{n-1}$$

Let $P_F = \alpha$, $\eta' = -\frac{1}{n}\log\alpha$, or equivalently

$$\tau' = \frac{\alpha^{-\frac{n-1}{n}} - 1}{n-1}$$

7. a) When $|y| > 1$, we decide H_1

When $|y| \le 1$,

$$L(y) = \frac{2-|y|}{4-4|y|} \ge \frac{1}{2} \quad \text{for } |y| \le 1$$

The likelihood threshold is $\dfrac{\pi_0}{2(1-\pi_0)}$

If $\dfrac{\pi_0}{2(1-\pi_0)} \le \dfrac{1}{2}$, i.e., $\pi_0 \le \dfrac{1}{2}$

H_1 is true,

$$V(\pi_0) = \pi_0 C_{10} p_0(\Gamma_1) + (1-\pi_0) C_{01} p_1(\Gamma_0) = \pi_0 C_{10}$$

If $\dfrac{\pi_0}{2(1-\pi_0)} > \dfrac{1}{2}$, i.e., $\pi_0 > \dfrac{1}{2}$

$$\delta_{\pi_0}(y) = \begin{cases} 1 & \text{if } |y| > \dfrac{4\pi_0 - 2}{3\pi_0 - 1} \\ 0 & 0 \le |y| \le \dfrac{4\pi_0 - 2}{3\pi_0 - 1} \end{cases}$$

$$V(\pi_0) = \pi_0 R_0(\delta_{\pi_0}) + \pi_1 R_1(\delta_{\pi_0})$$

$$R_0(\delta_{\pi_0}) = C_{10}\left(1 - \frac{4\pi_0 - 2}{3\pi_0 - 1}\right)^2$$

$$R_1(\delta_{\pi_0}) = C_{01}\left[\frac{4\pi_0 - 2}{3\pi_0 - 1} - \frac{1}{4}\left(\frac{4\pi_0 - 2}{3\pi_0 - 1}\right)^2\right]$$

Using $P_0(\delta_{\pi_L}) = P_1(\delta_{\pi_L})$

$$\pi_L = \frac{5+\sqrt{10}}{15},$$

the threshold $\tau = \dfrac{4\pi_L - 2}{3\pi_L - 1} = \dfrac{4-\sqrt{10}}{3}$

$$\delta(y) = \begin{cases} 1 & \text{if } |y| > \frac{4-\sqrt{10}}{3} \\ 0 & \text{if } |y| \leq \frac{4-\sqrt{10}}{3} \end{cases}$$

$$V(\pi_L) = C_{10}\left(\frac{\sqrt{10}-1}{3}\right)^2$$

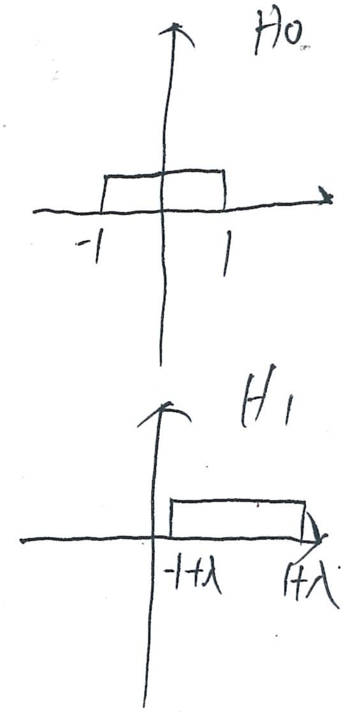
b) Let us define $\eta \geq 0$ as the threshold when we decide H_0 when $|y| \leq \eta$, then

$$P_F = (1-\eta)^2 = \alpha$$

which leads to $\eta = 1 - \sqrt{\alpha}$, and

$$P_D = \frac{(2-\eta)^2}{4} = \frac{(1+\sqrt{\alpha})^2}{4}$$

8. a) The pdf under H_0 and H_1 are shown in the right.



If $\alpha < \frac{1-(-1+\lambda)}{2} = \frac{2-\lambda}{2}$,

then the decision rule is

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } y \geq 1-2\alpha \\ 0 & \text{otherwise} \end{cases}$$

so that $P_F = \alpha$

If $\alpha \geq \frac{2-\lambda}{2}$,

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } y \geq -1+\lambda \\ 0 & \text{otherwise} \end{cases}$$

so that $P_F = \frac{2-\lambda}{2} < \alpha$

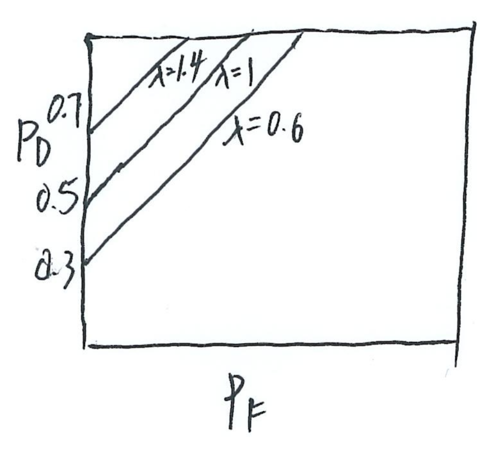
b) If $\alpha < \frac{2-\lambda}{2}$,

$$P_F = \alpha, \quad P_D = \frac{1}{2}(1+\lambda - (1-2\alpha))$$
$$= \frac{\lambda + 2\alpha}{2} = \alpha + \frac{\lambda}{2}$$

If $\alpha \geq \frac{2-\lambda}{2}$,

$$P_D = \frac{1}{2}(1+\lambda - (-1+\lambda)) = 1$$

The ROC curve can be sketched below



It can be seen that the larger λ results in better ROC since H_0 and H_1 are more separated.

9. The likelihood ratio is

$$L(y)=\frac{p_1(y)}{p_0(y)}=\frac{e^{-\frac{(y-1)^2}{2}}}{e^{-\frac{y^2}{2}}}=e^{y-\frac{1}{2}}$$

the threshold is

$$\tau=\frac{\pi_0(C_{10}-C_{00})}{\pi_1(C_{01}-C_{11})}$$

$$=\frac{\pi_0}{(1-\pi_0)N}=\frac{1}{N}$$

when N is large, $\tau\to 0$, the Bayes rule is

$\delta(y)=1$, where the risk is

$$r(\delta)=\pi_0 C_{10}\cdot p_0(\Gamma_1)=\frac{C_{10}}{2}=\frac{1}{2}$$

For the minimax test, let η denote the threshold that we decide H_1 when $y \geq \eta$, then

$$R_0(\delta) = C_{10}\,\phi(\eta)$$

$$R_1(\delta) = C_{01}\,\phi(1-\eta)$$

Letting $R_0(\delta) = R_1(\delta)$,

$$\phi(\eta) = N\,\phi(1-\eta)$$

$$\eta \to -\infty \quad \text{as} \quad N \to \infty,$$

and thus $\delta(y) = 1$, and the risk

$$r(\delta) = C_{10}\,P_0(\Gamma_1) = C_{10} = 1$$

We find that when N is very large, it is very costly to decide H_0 when H_1 is true. Therefore, both decision ~~rulse~~ rules will always decide H_1 to minimize the risk.