ENEE 621 Spring 2016 Homework 5 Solutions

1. Under H_0,

$$p(y|H_0) = \frac{p(y, H_0)}{p(H_0)}$$

$$= \frac{\int_0^\beta \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta}{\int_0^\beta \alpha e^{-\alpha\theta} d\theta}$$

Under H_1,

$$p(y|H_1) = \frac{p(y, H_1)}{p(H_1)}$$

$$= \frac{\int_\beta^\infty \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta}{\int_\beta^\infty \alpha e^{-\alpha\theta} d\theta}$$

The likelihood ratio can be written as

$$L(y) = \frac{p(y|H_1)}{p(y|H_0)}$$

$$= \frac{\int_\beta^\infty \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta}{\int_0^\beta \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta} \cdot \frac{\int_0^\beta \alpha e^{-\alpha\theta} d\theta}{\int_\beta^\infty \alpha e^{-\alpha\theta} d\theta}$$

The threshold is

$$\tau = \frac{\pi_0 (C_{10} - C_{00})}{\pi_1 (C_{01} - C_{11})} = \frac{p(H_0)}{p(H_1)} = \frac{\int_0^{\beta} \alpha e^{-\alpha\theta} d\theta}{\int_{\beta}^{\infty} \alpha e^{-\alpha\theta} d\theta}$$

Therefore, $L(y) \geq \tau \iff \dfrac{\int_{\beta}^{\infty} \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta}{\int_0^{\beta} \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta} \geq 1,$

which is equivalent to

$$\frac{e^{-\beta(\alpha+y)}((\alpha+y)\beta+1)}{1 - e^{-\beta(\alpha+y)}((\alpha+y)\beta+1)} \geq 1$$

Let y' be the solution of $e^{-\beta(\alpha+y)}((\alpha+y)\beta+1) = \frac{1}{2}$, the decision rule is

$$\delta_\beta(y) = \begin{cases} 1 & \text{if } y \leq y' \\ 0 & \text{if } y > y' \end{cases}$$

The Bayes risk is

$$r(\delta_B) = \pi_0 C_{10} p(\Gamma_1 | H_0) + \pi_1 C_{01} p(\Gamma_0 | H_1)$$

$$= C_{10} p(\Gamma_1, H_0) + C_{01} p(\Gamma_0, H_1)$$

$$= \int_0^{y'} \int_0^{B} \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta dy$$

$$+ \int_{y'}^{\infty} \int_{B}^{\infty} \theta e^{-\theta y} \alpha e^{-\alpha\theta} d\theta dy$$

2. Under H_0,

$$p(y|H_0) = \frac{\int_0^\beta \theta^n e^{-\theta \Sigma y_i} \alpha e^{-\alpha\theta} d\theta}{\int_0^\beta \alpha e^{-\alpha\theta} d\theta}$$

Under H_1,

$$p(y|H_1) = \frac{\int_\beta^\infty \theta^n e^{-\theta \Sigma y_i} \alpha e^{-\alpha\theta} d\theta}{\int_\beta^\infty \alpha e^{-\alpha\theta} d\theta}$$

$L(y) \geq \tau \iff F(y) \geq 1$ where

$$F(y) = \frac{\int_\beta^\infty \theta^n e^{-\theta \Sigma y_i} \alpha e^{-\alpha\theta} d\theta}{\int_0^\beta \theta^n e^{-\theta \Sigma y_i} \alpha e^{-\alpha\theta} d\theta}$$

The likelihood ratio test can be written as

$$\delta_B(y) = \begin{cases} 1 & \text{if } F(y) \geq 1 \\ 0 & \text{if } F(y) < 1 \end{cases}$$

The Bayes risk can be written as

$$r(\delta_\beta) = \pi_0 C_{10} P(\Gamma_1 | H_0) + \pi_1 C_{01} P(\Gamma_0 | H_1)$$

$$= \int_{\{y: F(y) \geq 1\}} \int_0^\beta \theta e^{-\theta y} \alpha e^{-\alpha\theta} \, d\theta \, dy$$

$$+ \int_{\{y: F(y) < 1\}} \int_\beta^\infty \theta e^{-\theta y} \alpha e^{-\alpha\theta} \, d\theta \, dy$$

3. a) The likelihood ratio can be written as

$$L(y) = \frac{\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-(y_k-\mu_1)^2/2\sigma_1^2}}{\prod_{k=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(y_k-\mu_0)^2/2\sigma_0^2}}$$

$$= \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{\frac{n}{2}\left(\frac{\mu_0^2}{\sigma_0^2} - \frac{\mu_1^2}{\sigma_1^2}\right)} e^{\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)\sum_{k=1}^{n} y_k^2} .$$

$$e^{\left(\frac{\mu_1}{\sigma_1^2} - \frac{\mu_0}{\sigma_0^2}\right)\sum_{k=1}^{n} y_k} ,$$

which is of the desired form

b) If $\mu_0 = \mu_1$, $\sigma_1^2 > \sigma_0^2$,

$$L(y) = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{\sum_{k=1}^{n} (y_k-\mu)^2 \left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)}$$

$$L(y) \geq \tau \iff \sum_{k=1}^{n} (y_k-\mu)^2 \geq \tau'$$

In otherwords, we compare $\sum_{k=1}^{n} (y_k-\mu)^2$ with some threshold.

If $\sigma_0^2 = \sigma_1^2$, $u_1 > u_0$,

$$L(y) = e^{\sum_{i=1}^{k} (y_k - u_0)^2 - \sum_{i=1}^{k} (y_k - u_1)^2}$$

$$= e^{2(u_1 - u_0) \sum_{i=1}^{n} y_k} \cdot e^{n(u_0^2 - u_1^2)}$$

$$L(y) \geq \tau \iff \sum_{i=1}^{n} y_k \geq \tau''$$

In other words, we compare $\sum_{i=1}^{n} y_k$ with some threshold.

c) For $n=1$, $u_1 = u_0 = u$ and $\sigma_1^2 > \sigma_0^2$, the NP test is

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } (y-u)^2 \geq \tau' \\ 0 & \text{if } (y-u)^2 < \tau' \end{cases}$$

the false alarm probability is

$$P_F(\delta_{NP}) = \cancel{P_0(\ldots)} \; P_0((y-u)^2 \geq \tau')$$

$$= 2\left[1 - \phi\left(\frac{\sqrt{\tau'}}{\sigma_0}\right)\right]$$

where $\phi(\cdot)$ is the CDF of normal distribution.

Therefore, the appropriate τ' for α is

$$\tau' = \left[\sigma_0 \phi^{-1} (1 - \frac{\alpha}{2}) \right]^2$$

The detection probability is

$$P_D(\delta_{NP}) = 1 - P_1((y-\mu)^2 < \tau')$$

$$= 2\left[1 - \phi\left(\frac{\sigma_0}{\sigma_1} \phi^{-1}(1 - \frac{\alpha}{2})\right)\right], \quad 0 < \alpha < 1$$

4. If $u = u_1 > u_0 = 0$ and $\sigma^2 = \sigma_0^2 = \sigma_1^2 > 0$,

$$L(y) = e^{2u \sum_{i=1}^{n} y_k} e^{-nu^2}$$

The NP test is

$$\delta_{NP}(y) = \begin{cases} 1 & \text{if } \sum y_i \geq \tau'' \\ 0 & \text{if } \sum y_i < \tau'' \end{cases}$$

$$P_F = P_0 \left(\sum y_i \geq \tau'' \right)$$

$$= 1 - \phi\left(\frac{\tau''}{\sqrt{n}\,\sigma} \right)$$

therefore, the appropriate τ'' for α is

$$\tau'' = \phi^{-1}(1-\alpha)\sqrt{n} \cdot \sigma$$

If σ is unknown, there does not exist a UMP for any σ.

$$P_D = P_1 \left(\sum y_i \geq \tau'' \right)$$

$$= 1 - \phi\left(\frac{\tau'' - nu}{\sqrt{n}\,\sigma} \right)$$

5. a) From chapter 2.5 of the lecture notes, with the error probability criteria, the decision becomes ~~on~~ a MAP computer.

Under equal priors, it reduces to a maximum likelihood problem.

The decision rule is

$$\delta(y) = \begin{cases} 1 & \text{if } y > 1 \\ 2 & \text{if } y < -1 \\ r_1 & \text{if } y \in [0, 1] \\ r_2 & \text{if } y \in [-1, 0) \end{cases}$$

where r_1 can be 0 or 1 with any probability,
r_2 can be 0 or 2 with any probability,
without changing the error probability.

b) Take $r_1 = 0$ and $r_2 = 0$ as an example, the error probability is $\frac{1}{3} \times P_0(\Gamma_1 \cup \Gamma_2) + \frac{1}{3} P_1(\Gamma_0 \cup \Gamma_2) + \frac{1}{3} P_2(\Gamma_0 \cup \Gamma_1)$

$$= 0 + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}$$

$$= \frac{1}{3}$$