ENEE 621 Spring 2016 Homework 6 Solutions

1. a) In order to find the MAP estimate, we are to find the

$\hat{\theta}_{MAP}(y)$ that maximizes $p(\theta|y) = \dfrac{p(\theta,y)}{p(y)}$

Since $p(y)$ is the same for the same y, we need to find $\hat{\theta}_{MAP}(y)$ that maximizes $p(\theta,y)$

$$p(\theta,y) = \frac{1}{\theta} \cdot \frac{\theta}{2} e^{-\theta|y|} = \frac{1}{2} e^{-\theta|y|}$$

therefore $\hat{\theta}_{MAP}(y) = 1$

b) $\hat{\theta}_{MMSE}(y) = E[\theta|y]$

$$p(\theta|y) = \frac{p(\theta,y)}{p(y)} = \frac{\frac{1}{2} e^{-\theta|y|}}{\int_1^e \frac{1}{2} e^{-\theta|y|}\, d\theta} = \frac{e^{-\theta|y|}}{\int_1^e e^{-\theta|y|}\, d\theta}$$

$$E[\theta|y] = \int_1^e \theta\, p(\theta|y)\, d\theta = \frac{\int_1^e \theta e^{-\theta|y|}\, d\theta}{\int_1^e e^{-\theta|y|}\, d\theta}$$

$$= \frac{1}{|y|} + \frac{e^{-|y|} - e^{1-e|y|}}{e^{-|y|} - e^{-e|y|}}$$

2. a) $p(\theta, y) = w(\theta)\, p(y|\theta)$

$p(y|\theta) = P(N = y-\theta) \cdot P(S=1) + P(N = y+\theta) P(S=-1)$

$= \frac{1}{2}\left(f(y+\theta) + f(y-\theta)\right)$

where f is the pdf function of normal distribution

$\hat{\theta}_{MMSE}(y) = E(\theta|y) = \int_0^1 \theta\, p(\theta|y)\, d\theta$

$$p(\theta|y) = \frac{p(\theta, y)}{p(y)} = \frac{\frac{k}{2} e^{\frac{\theta^2}{2}} \left(f(y+\theta) + f(y-\theta)\right)}{\int_0^1 \frac{k}{2} e^{-\frac{\theta^2}{2}} \left(f(y+\theta) + f(y-\theta)\right) d\theta}$$

$$\hat{\theta}_{MMSE}(y) = \frac{\int_0^1 \theta e^{\frac{\theta^2}{2}} \left(f(y+\theta) + f(y-\theta)\right) d\theta}{\int_0^1 e^{\frac{\theta^2}{2}} \left(f(y+\theta) + f(y-\theta)\right) d\theta}$$

b) $p(\theta, y) = w(\theta)\, p(y|\theta) = K e^{\frac{\theta^2}{2}} \cdot \frac{1}{2}\left(f(y+\theta) + f(y-\theta)\right)$

$= \frac{K}{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \left(e^{y\theta} + e^{-y\theta}\right)$

The $\theta \in [0,1]$ that maximizes $p(\theta, y)$ is

$\hat{\theta}_{MAP}(y) = 1$

3. $$p(\theta|y) = \frac{\theta^y e^{-\theta} e^{-\alpha\theta}}{\int_0^\infty \theta^y e^{-\theta} e^{-\alpha\theta} d\theta} = \frac{\theta^y e^{-(\alpha+1)\theta} (1+\alpha)^{y+1}}{y!}$$

Therefore,

$$\hat{\theta}_{MMSE}(y) = \frac{1}{y!} \int_0^\infty \theta^{y+1} e^{-(\alpha+1)\theta} d\theta \, (1+\alpha)^{y+1} = \frac{y+1}{\alpha+1}$$

$$\hat{\theta}_{MAP}(y) = \arg\max_{\theta>0} \left(y \log\theta - (\alpha+1)\theta \right) = \frac{y}{\alpha+1}$$

the MMAE satisfies

$$\int_0^{\hat{\theta}_{MMAE}(y)} p(\theta|y) \, d\theta = \frac{1}{2},$$

which can be obtained by numerical methods.

4. a) $p(y|\theta) = f(y-\theta)$

where f is the pdf function of the Gaussian distribution in the problem.

$$p(\theta|y) = \frac{p(y,\theta)}{p(y)} = \begin{cases} \dfrac{f(y-\theta)\cdot\frac{1}{2}}{(f(y-1)+f(y+1))\cdot\frac{1}{2}} & \text{if } \theta=1 \text{ or } \theta=-1. \\ 0 & \text{otherwise.} \end{cases}$$

therefore, $\hat{\theta}_{MAP}(y) = \arg\max_{\theta} f(y-\theta)$

$$= \begin{cases} 1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0 \end{cases}$$

$$\hat{\theta}_{MMSE}(y) = E[\theta|y] = \frac{f(y-1)}{f(y-1)+f(y+1)}\cdot 1 + \frac{f(y+1)}{f(y-1)+f(y+1)}\cdot(-1)$$

$$= \frac{f(y-1)-f(y+1)}{f(y-1)+f(y+1)}$$

b) The MMSE estimate can be written as

$$\hat{\theta}_{MMSE}(y) = \frac{e^{-\frac{(y-1)^2}{2\sigma^2}} - e^{-\frac{(y+1)^2}{2\sigma^2}}}{e^{-\frac{(y-1)^2}{2\sigma^2}} + e^{-\frac{(y+1)^2}{2\sigma^2}}}$$

If σ^2 is very small,

$e^{-\frac{(y-1)^2}{2\sigma^2}}$ will dominate $e^{-\frac{(y+1)^2}{2\sigma^2}}$ when $y>0$,

$e^{-\frac{(y+1)^2}{2\sigma^2}}$ will dominate $e^{-\frac{(y-1)^2}{2\sigma^2}}$ when $y<0$,

the MAP and MMSE estimates are approximately equal.

5. $P(Y=y|\theta) = \binom{m}{y}\theta^{y}(1-\theta)^{m-y}$

$$P(Y=y+1|\theta) = \binom{m}{y+1}\theta^{y+1}(1-\theta)^{m-y-1}\, d\theta$$

Since $\theta \sim U(0,1)$,

$$P(Y=y) = \int_0^1 p(Y=y|\theta)\, d\theta$$

$$= \int_0^1 \binom{m}{y}\theta^{y}(1-\theta)^{m-y}\, d\theta$$

$$= \binom{m}{y}\frac{\Gamma(y+1)\,\Gamma(m-y+1)}{\Gamma(m+2)}$$

where $\Gamma(.)$ is the gamma function,

$\Gamma(n) = (n-1)!$ if n is positive integer.

Similarly, $P(Y=y+1) = \binom{m}{y+1}\dfrac{\Gamma(y+2)\,\Gamma(m-y)}{\Gamma(m+2)}$

It is easy to verify that $P(Y=y+1) = P(Y=y)$

6. $$p_\theta(y) = \begin{cases} \dfrac{1}{\beta-\alpha} & \text{if } y \in (\alpha, \beta) \\ 0 & \text{otherwise} \end{cases}$$

$p_\theta(y)$ can be written as

$$p_\theta(y) = \frac{1}{\beta-\alpha} \cdot I(y \in (\alpha, \beta)),$$

which can not be written into the form in (IV.1.8) since $I(y \in (\alpha, \beta))$ depends on both θ and y.

Therefore, it's not an exponential family

7. $E_\theta[f(y)] = \int_\alpha^\beta \frac{1}{\beta-\alpha} f(y)\, dy$

$$= \frac{1}{\beta-\alpha} \int_\alpha^\beta f(y)\, dy$$

To satisfy that $E_\theta[f(y)] = 0$,

$f(y) = 0$ for $y \in [0,1]$,

and the family is a complete family

8. Since $p_\theta(y) = (1-\theta)\theta^y, \quad y=0,1\cdots$

$$E_\theta[f(y)] = \sum_{y=0}^{\infty}(1-\theta)\theta^y f(y)$$

$$= (1-\theta)\sum_{y=0}^{\infty}\theta^y f(y)$$

As $\theta \in (0,1)$, $\theta^y > 0$ for $y=0,1\cdots$

In order to make $E_\theta[f(y)] = 0$,

$f(y) = 0$ for $y=0,1\cdots$

Therefore the family is complete

9. First note that Y_i's are not independent since they are related through the term θX.

Let $f(\cdot)$ denote the pdf of normal distribution,

$$p_\theta(Y) = \int_{-\infty}^{\infty} p(X=x) \cdot p(V_1 = Y_1 - \theta x, V_2 = Y_2 - \theta x, \cdots, V_k = Y_k - \theta x)\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot \prod_{i=1}^{k} f(Y_i - \theta x)\, dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{k+1} \int_{-\infty}^{\infty} e^{-\frac{x^2 + k\theta^2 x^2 - 2\theta \sum Y_i x + \sum Y_i^2}{2}}\, dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{k+1} \int_{-\infty}^{\infty} e^{-\frac{(k\theta^2+1)x^2 - 2\theta \sum Y_i x}{2}}\, dx \cdot e^{-\frac{\sum Y_i^2}{2}}$$

By the factorization theorem,

$T(Y) = \sum Y_i$ is the sufficient statistics.

10. $p_\theta(Y)$ can be farther written as

$$p_\theta(Y) = \left(\frac{1}{\sqrt{2\pi}}\right)^{k+1} e^{-\frac{\Sigma Y_i^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(k\theta^2+1)x^2 - 2\theta \Sigma Y_i x}{2}} dx$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^{k+1} e^{-\frac{\Sigma Y_i^2}{2}} \frac{\sqrt{\pi}\, e^{\frac{(\theta \Sigma Y_i)^2}{2(k\theta^2+1)}}}{\sqrt{\frac{k\theta^2+1}{2}}}$$

which can be written in the form of (IV. C. 8) in the textbook, and the family is exponential family.