ENEE 621 2016 Homework 7 Solutions

1. $p(\theta,y) = \begin{cases} \frac{1}{2} e^{-|y-\theta|} \cdot e^{-\theta} & \theta \geq 0 \\ 0 & \theta < 0 \end{cases}$

If $y > 0$,

$$p(\theta,y) = \begin{cases} \frac{1}{2} e^{-y} & 0 \leq \theta < y \\ \frac{1}{2} e^{-2\theta+y} & \theta \geq y \end{cases}$$

$\hat{\theta}_{MAP}(y)$ can be any value in $[0, y]$

If $y \leq 0$,

$$p(\theta,y) = \frac{1}{2} e^{-2\theta+y}, \quad \hat{\theta}_{MAP}(y) = 0$$

In order to calculate the MMSE estimate, we need to have $\cancel{p(y|\theta)}$ $p(\theta|y) = \frac{p(\theta,y)}{p(y)}$

$$p(y) = \int_0^{+\infty} p(\theta,y)\, d\theta$$

If $y > 0$, $p(y) = \int_0^{y} \frac{1}{2} e^{-y} d\theta + \int_y^{\infty} \frac{1}{2} e^{-2\theta+y} d\theta$

$$= \frac{1}{2} e^{-y} \cdot y + \frac{1}{4} e^{-y}$$

If $y \le 0$, $p(y) = \int_0^{\infty} \frac{1}{2} e^{-2\theta+y} \, d\theta$

$$= \frac{1}{4} e^{y}$$

Therefore, we are ready to calculate $\hat{\theta}_{MMSE}(y)$

$$\hat{\theta}_{MMSE}(y) = E[\theta | y]$$

$$= \int_0^{\infty} \theta \, p(\theta | y) \, d\theta$$

If $y > 0$,

$$\hat{\theta}_{MMSE}(y) = \int_0^{y} \frac{\frac{1}{2} e^{-y} \theta}{\frac{1}{2} e^{-y} \cdot y + \frac{1}{4} e^{-y}} \, d\theta + \int_y^{\infty} \frac{\frac{1}{2} e^{-2\theta+y} \, \theta}{\frac{1}{2} e^{-y} y + \frac{1}{4} e^{-y}} \, d\theta$$

$$= \int_0^{y} \frac{2\theta}{2y+1} \, d\theta + \int_y^{\infty} \frac{2\theta e^{-2\theta+y}}{2e^{-y} y + e^{-y}} \, d\theta$$

If $y \le 0$

$$\hat{\theta}_{MMSE}(y) = \int_0^{\infty} \frac{\frac{1}{2} e^{-2\theta+y} \cdot \theta}{\frac{1}{4} e^{y}} \, d\theta$$

$$= \int_0^{\infty} 2\theta e^{-2\theta} \, d\theta$$

2. First, we notice that Y_1 and Y_2 are not independent since N_1 and N_2 are not independent

$$f(n_1, n_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(n_1^2 + n_2^2 - 2\rho n_1 n_2\right)\right)$$

a)
$$p(\theta, y) = \begin{cases} \frac{1}{\alpha} f(y_1\sqrt{\theta}, y_2\sqrt{\theta}) & \theta \in [0, \alpha] \\ 0 & \text{otherwise} \end{cases}$$

$$p(\theta|y) = \frac{p(\theta, y)}{p(y)} = \frac{f(y_1\sqrt{\theta}, y_2\sqrt{\theta})}{\int_0^\alpha f(y_1\sqrt{\theta}, y_2\sqrt{\theta})\, d\theta}$$

$$\hat{\theta}_{MMSE}(y) = E[\theta|y] = \frac{\int_0^\alpha f(y_1\sqrt{\theta}, y_2\sqrt{\theta})\, \theta\, d\theta}{\int_0^\alpha f(y_1\sqrt{\theta}, y_2\sqrt{\theta})\, d\theta}$$

b) $p(\theta, y)$ can be written as

$$p(\theta, y) = \begin{cases} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{\theta}{2(1-\rho^2)}\left(y_1^2 + y_2^2 - 2\rho y_1 y_2\right)\right) & \text{if } \theta \in [0, \alpha] \\ 0 & \text{otherwise} \end{cases}$$

Since $y_1^2 + y_2^2 - 2\rho y_1 y_2 \geq (\sqrt{\rho} y_1 - \sqrt{\rho} y_2)^2 \geq 0$,
in order to maximize $p(\theta, y)$, $\hat{\theta}_{MAP}(y) = 0$

c) The MMAE estimation $\hat{\theta}_{MMAE}(y)$ should satisfy

$$\int_{-\infty}^{\hat{\theta}_{MMAE}} p(\theta|y)\, d\theta = \frac{1}{2}.$$

Therefore, $\hat{\theta}_{MMAE}(y)$ is the solution to the equation

$$\frac{\int_0^{\hat{\theta}_{MMAE}} f(y_1\sqrt{\theta}, y_2\sqrt{\theta})\, d\theta}{\int_0^{\alpha} f(y_1\sqrt{\theta}; y_2\sqrt{\theta})\, d\theta} = \frac{1}{2}$$

3. a) We have $p(\theta, y) = \begin{cases} e^{-y+\theta} e^{-\theta} & 0 \le \theta \le y \\ 0 & \text{otherwise.} \end{cases}$

$$p(y) = e^{-y} \int_0^y d\theta = y e^{-y}, \quad y > 0.$$

$$p(\theta | y) = \frac{p(\theta, y)}{p(y)} = \frac{1}{y} \quad \text{for } 0 \le \theta \le y.$$

In other words, θ is uniformly distributed in the interval $[0, y]$ given y. Therefore,

$$\hat{\theta}_{MMSE}(y) = \hat{\theta}_{MMAE}(y) = \frac{y}{2}.$$

b) Since $p(\theta|y)$ is uniform on $[0, y]$, $Var(\theta|y) = \frac{y^2}{12}$

$$MMSE = E_y[Var(\theta|y)] = E_y\left[\frac{y^2}{12}\right]$$

$$= \int_0^\infty p(y) \cdot \frac{y^2}{12} dy$$

$$= \int_0^\infty \frac{y^3}{12} e^{-y} dy$$

$$= \frac{1}{2}$$

c) We can have

$$p(\theta, y) = \begin{cases} \prod_{k=1}^{n} e^{-y_k + \theta} & \text{if } 0 < \theta < \min\{y_1 \cdots y_n\} \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that $\hat{\theta}_{MAP}(y) = \min\{y_1 \cdots y_n\}$.

4. Let $f(\cdot)$ denote the pdf function shared by $N_1 \cdots N_n$,
$g(\cdot)$ denote the pdf function of θ.

$$p(\theta, y) = \prod_{i=1}^{n} f(y_i - \theta \alpha^i) \cdot g(\theta)$$

$$p(y) = \int_{-\infty}^{\infty} p(\theta, y)\, d\theta$$
$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} f(y_i - \theta\alpha^i)\, g(\theta)\, d\theta.$$

$$\hat{\theta}_{MMSE}(y) = \cancel{E[y|\theta]}\ E[\theta|y]$$
$$= \frac{\int_{-\infty}^{\infty} \prod_{i=1}^{n} f(y_i - \theta\alpha^i)\, g(\theta) \cdot \theta\, d\theta}{\int_{-\infty}^{\infty} \prod_{i=1}^{n} f(y_i - \theta\alpha^i)\, g(\theta)\, d\theta}$$

5. $$p_\theta(y) = \begin{cases} (\theta-1)^n \prod_{k=1}^{n} y_k^{-\theta} & \text{if } \min(y_1 \cdots y_n) \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (\theta-1)^n \left(\prod_{k=1}^{n} y_k\right)^{-\theta} & \text{if } \min(y_1 \cdots y_n) \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, by the Factorization Theorem (IV.C.2) in the textbook, we can write

$$p_\theta(y) = g_\theta[T(y)]\, h(y)$$

$$T(y) = \prod_{k=1}^{n} y_k, \quad g_\theta(t) = (\theta-1)^n t^{-n},$$

$$h(y) = I(\min(y_1 \cdots y_n) \geq 1) = \begin{cases} 1 & \text{if } \min(y_1 \cdots y_n) \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

6. a) We have $p_\theta(y) = \theta^{T(y)} (1-\theta)^{(n-T(y))}$

where $T(y) = \sum_{k=1}^{n} y_k$

Rewriting this as $p_\theta(y) = C(\phi)\, e^{\phi T(y)}$

with $\phi = \log\left(\frac{\theta}{1-\theta}\right)$ and $C(\phi) = e^{n\phi}$.

We see from the Completeness Theorem for Exponential Families that $T(y)$ is a complete sufficient statistic for ϕ and hence for θ.

Thus, any unbiased function of $T(y)$ is an MVUE.

$E_\theta[T(y)] = n\theta$, the MVUE is given by

$$\hat{\theta}_{MVUE}(y) = \frac{T(y)}{n}$$

b)

$$\hat{\theta}_{ML}(y) = \arg\max_\theta \theta^{T(y)} (1-\theta)^{n-T(y)}$$

$$= \frac{T(y)}{n} = \hat{\theta}_{MVUE}(y)$$

$E_\theta[\hat{\theta}_{ML}(y)] = \theta$, $\quad Var[\hat{\theta}_{ML}(y)] = \frac{\theta(1-\theta)}{n}$

c) $\frac{\partial^2}{\partial \theta^2} \log p_\theta(y) = -\frac{T(y)}{\theta^2} - \frac{n-T(y)}{(1-\theta)^2}$

$$I_\theta = -E\left[-\frac{T(y)}{\theta^2} - \frac{n-T(y)}{(1-\theta)^2}\right] = \frac{n}{\theta} + \frac{n}{1-\theta}$$

$$= \frac{n}{\theta(1-\theta)}$$

The CRLB is

$$CRLB = \frac{1}{I_\theta} = \frac{\theta(1-\theta)}{n},$$

which is equivalent to the variance obtained by the MVUE and ML.

7. We have $p_\theta(y) = \dfrac{e^{-\theta}\theta^y}{y!}$, $y = 0, 1, \cdots$

$$\frac{\partial}{\partial\theta}\log p_\theta(y) = \frac{\partial}{\partial\theta}(-\theta + y\log\theta - \log y!) = -1 + \frac{y}{\theta}$$

and $\dfrac{\partial^2}{\partial\theta^2}\log p_\theta(y) = -\dfrac{y}{\theta^2} < 0$

Thus, $\hat{\theta}_{ML}(y) = y$.

According to the statistics of Poisson distribution,

$$E_\theta[\hat{\theta}_{ML}(y)] = Var_\theta[\hat{\theta}_{ML}(y)] = \theta,$$

so the maximum likelihood estimation is unbiased.

$$I_\theta = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log p_\theta(y)\right] = \frac{E_\theta[y]}{\theta^2} = \frac{1}{\theta},$$

CRLB $= \theta$, which equals $Var_\theta[\hat{\theta}_{ML}(y)]$.

8. a) With $y_1 \cdots y_n$, $f_\theta(y)$ can be written as

$$f_\theta(y) = \begin{cases} \dfrac{\prod_{i=1}^{n} y_i^{M} e^{-\frac{\sum_{i=1}^{n} y_i}{2\theta}}}{\left((2\theta)^{M+1} M!\right)^n} & y_1 \cdots y_n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\log f_\theta(y) = M \sum_{i=1}^{n} \log y_i - \frac{1}{2\theta} \sum_{i=1}^{n} y_i - n(M+1)\log(2\theta) - n\log(M!)$$

$$\frac{\partial}{\partial\theta} \log f_\theta(y) = \frac{1}{2\theta^2} \sum_{i=1}^{n} y_i - \frac{1}{\theta} \cdot n(M+1)$$

$$\hat{\theta}_{ML}(y) = \frac{\sum_{i=1}^{n} y_i}{2n(M+1)}$$

b) $$E[y_i] = \int_0^\infty \frac{y^{M+1} e^{-\frac{y}{2\theta}} \cancel{d\theta}}{(2\theta)^{M+1} M!} \cancel{d\theta}\, dy = 2\theta(M+1)$$

$$E[y_i^2] = \int_0^\infty \frac{y^{M+2} e^{-\frac{y}{2\theta}}}{(2\theta)^{M+1} M!} \cancel{d\theta}\, dy\ 4\theta^2 (M+1)(M+2)$$

$E[\hat{\theta}_{ML}(y)] = \theta$, the bias is 0

To calculate $Var[\hat{\theta}_{ML}(y)]$, notice that

$$Var\left[\frac{y_i}{2(M+1)}\right] = \frac{1}{4(M+1)^2}\left(E[y_i^2] - E[y_i]^2\right)$$

$$= \frac{\theta^2}{M+1}$$

$$Var[\hat{\theta}_{ML}(y)] = \frac{\theta^2}{n(M+1)}$$

c) $$\frac{\partial}{\partial\theta^2}\log f_\theta(y) = -\frac{\Sigma y_i}{\theta^3} + \frac{n(M+1)}{\theta^2}$$

$$I_\theta = -E_\theta\left[\frac{\partial}{\partial\theta^2}\log f_\theta(y)\right] = \frac{n(M+1)}{\theta^2}$$

$$CRLB = \frac{1}{I_\theta} = \frac{\theta^2}{n(M+1)}$$

d) Since it's unbiased, it's consistent.

Since it achieves CRLB, it's efficient.

9. a) $g_1(y_1 \cdots y_k) = \frac{1}{k}\sum_{l=1}^{k} y_l$

$E[g_1(y_1 \cdots y_k)] = \frac{1}{k}\sum_{l=1}^{k} E[y_l] = m.$

$g_1(y_1 \cdots y_k)$ is unbiased estimator of m.

b). Let g_1 denote $g_1(y_1 \cdots y_k)$.

$$\sum_{l=1}^{k} (y_l - g_1)^2$$

$$= \sum_{l=1}^{k} y_l^2 - 2g_1 \sum_{l=1}^{k} y_l + kg_1^2$$

$$= \sum_{l=1}^{k} y_l^2 - 2k g_1^2 + kg_1^2$$

$$= \sum_{l=1}^{k} y_l^2 - kg_1^2$$

$$E[\sum_{l=1}^{k} (y_l - g_1)^2] = E[\sum_{l=1}^{k} y_l^2] - kE[g_1^2]$$

We know that $E[y_i^2] = m^2 + \sigma^2$ for $i = 1, 2 \cdots k$.

$$E[g_1^2] = \frac{\sigma^2}{k} + m^2$$

Thus,

$$E\left[\sum_{i=1}^{k}(y_i-g_1)^2\right]$$

$$=(k-1)\sigma^2$$

$$E[g_2(y_1\cdots y_k)]=\frac{k-1}{k}\sigma^2\neq\sigma^2,$$

g_2 is not an unbiased estimator of σ^2

10. $p_\theta(y) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!}$

$$= \frac{e^{-\theta n}\,\theta^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}$$

$$= e^{-\theta n} \exp\{\log\theta \cdot \sum_{i=1}^{n} y_i\} \cdot \frac{1}{\prod_{i=1}^{n} y_i!}$$

Let $\phi = \log\theta$, $p_\theta(y) = e^{-ne^{\phi}} \exp\{\phi\, T(y_1, \cdots y_n)\} \cdot \frac{1}{\prod_{i=1}^{n} y_i!}$

By the completeness theorem for Exponential Families,

$T(y)$ is a complete sufficient statistics for the family $\{F_\theta^{(n)}, \theta > 0\}$.

Since $E[T(y)] = n\theta$, the MVUE can be found as

$$\hat{\theta}_{MVUE}(y) = \frac{T(y)}{n}$$