# ESTIMATION AND DETECTION THEORY

## HOMEWORK # 8:

Please work out the **ten** (10) problems stated below – HVP refers to the text: H. Vincent Poor, *An Introduction to Signal Detection and Estimation* (Second Edition), Springer Texts in Electrical Engineering Springer, New York (NY), 2010. With this in mind, Exercise **II.2** (HVP) refers to Exercise 2 for Chapter II of HVP. Exercises are located at the end of each chapter.

**Show** work and **explain** reasoning.

**1.**

Solve Exercise **IV.5** (HVP).

**2.**

Assume the family $\{F_\theta,\ \theta \in \Theta\}$ to be an exponential family (with respect to some distribution function $F$ on $\mathbb{R}^k$) with density functions of the form

$$f_\theta(\boldsymbol{y}) = C(\theta)q(\boldsymbol{y})e^{Q(\theta)'K(\boldsymbol{y})} \quad F - \text{a.e.}$$

for every $\theta$ in $\Theta$ with Borel mappings $C : \Theta \to \mathbb{R}_+$, $Q : \Theta \to \mathbb{R}^q$, $q : \mathbb{R}^k \to \mathbb{R}_+$, and $K : \mathbb{R}^k \to \mathbb{R}^q$.

**2.a** Specialize Conditions **(CR1)**–**(CR5)** in terms (of properties) of the Borel mappings entering the definition of the exponential family. In particular, show that the regularity condition **(CR5)** is equivalent to

$$\left(\frac{\partial}{\partial\theta_i}Q(\theta)\right)' \mathbb{E}_\theta\left[K(\boldsymbol{Y})\right] = -\frac{\partial}{\partial\theta_i}\log C(\theta), \quad \begin{matrix}\theta_i \in \Theta \\ i = 1, \ldots, p.\end{matrix}$$

**2.b** Find an expression for the Fisher information matrix $M(\theta)$ in terms of the covariance $\text{Cov}_\theta\left[K(\boldsymbol{Y})\right]$. **HINT:** Use Part **2.a** together with the fact that

$$\frac{\partial}{\partial\theta_i}\log f_\theta(\boldsymbol{y}) = \frac{\partial}{\partial\theta_i}\log C(\theta) + \frac{\partial}{\partial\theta_i}Q(\theta)'K(\boldsymbol{y}), \qquad \begin{matrix} i = 1, \ldots, p \\ \boldsymbol{y} \in S\end{matrix}$$

**2.c** What are the conditions that need to hold for an estimator $g : \mathbb{R}^k \to \mathbb{R}^p$ to be a regular estimator?

**3.**

Solve Exercise **IV.15** (HVP).

**4.**

Solve Exercise **IV.21** Part (a) (HVP).

**5.**

Solve Exercise **IV.22** (HVP).

**6.**

In the context of the Cramèr-Rao bounds, with arbitrary $p$, consider the $\mathbb{R}^p$-valued rv $U(\theta, \boldsymbol{Y})$ given by

$$U(\theta, \boldsymbol{Y}) = g(\boldsymbol{Y}) - \theta - b_\theta(g) - (\boldsymbol{I}_p + \nabla_\theta b_\theta(g))\, M(\theta)^{-1} \nabla_\theta \log f_\theta(\boldsymbol{Y}), \quad \theta \in \Theta.$$

Show that the rv $U(\theta, \boldsymbol{Y})$ has zero mean and that its covariance matrix is given by

$$\begin{aligned} \mathrm{Cov}_\theta[U(\theta, \boldsymbol{Y})] &= \mathrm{Cov}_\theta\left[U(\theta, \boldsymbol{Y})\right] \\ &= \Sigma_\theta(g) - b_\theta(g) b_\theta(g)' \\ &\quad - (\boldsymbol{I}_p + \nabla_\theta b_\theta(g))\, M(\theta)^{-1} \left(\boldsymbol{I}_p + \nabla_\theta b_\theta(g)\right)'. \end{aligned} \tag{1.1}$$

Use this fact to show that the Cramèr-Rao bound is equivalent to the statement that the covariance matrix $\mathrm{Cov}_\theta[U(\theta, \boldsymbol{Y})]$ is positive semi-definite. Explore what happens when the bound is achieved.

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In Exercises **7** to **9**, a family of distributions $\{F_\theta,\ \theta \in \Theta\}$ is given. For each $n = 1, 2, \ldots,$, let $\{F_\theta^{(n)},\ \theta \in \Theta\}$ denote the corresponding families associated with $n$ i.i.d. samples. In each case, (i) compute the Fisher information matrices $M^{(n)}(\theta)$ for each $\theta$ in $\Theta$; (ii) find the efficient estimator for $\theta$ on the basis of the samples $Y_1, \ldots, Y_n$; (iii) find the ML estimator of $\theta$ on the samples $Y_1, \ldots, Y_n$; (iv) Are these estimators unbiased? asymptotically unbiased? consistent? asymptotically normal?

**7.**

Here $\Theta = (0, \infty)$ and for each $\theta > 0$, $F_\theta$ is an exponential distribution with parameter $\theta$, namely

$$F_\theta(y) = \left(1 - e^{-\theta y}\right)^+, \quad y \in \mathbb{R}.$$

**8.**

Here $\Theta = \mathbb{R}$ and for each $\theta > 0$, $F_\theta$ is the shifted Cauchy distribution whose probability density function is given by

$$f_\theta(y) = \frac{1}{\pi(1 + (y - \theta)^2)}, \quad y \in \mathbb{R}.$$

**9.**

Here $\Theta = (0, \infty)$ and for each $\theta > 0$, $F_\theta$ is the Gaussian distribution with zero mean and variance $\theta$.

**10.**

Solve Exercise **IV.23** (HVP).

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