ENEE 630 Homework 4[1]

Material Covered: Multirate Maximally/Non-maximally Decimated System, Alias Free, Distortion Function, AC matrix, P matrix

Problem 1 Suppose the M-channel maximally decimated QMF bank is alias-free, and let T(z) be the distortion function. Suppose we define a new filter bank in which the analysis and synthesis filters are interchanged, that is $F_k(z)$ are the analysis filters and $H_k(z)$ are the synthesis filters. Show that the resulting system is free from aliasing and has the same distortion function T(z). So we can swap each $F_k(z)$ with the corresponding $H_k(z)$ without changing the input/output properties.
(Hint: Use AC matrix formulation cleverly.)

Problem 2 Consider Fig. P-2 with $T = W^*$ (a uniform DFT analysis bank). Suppose $R_k(z)$ are chosen as in Eq (1.1) below, so that the product $R_k(z)E_k(z)$ is independent of k. This ensures that aliasing has been cancelled.

$$R_k(z) = \prod_{l \neq k} E_l(z) \qquad (1.1)$$

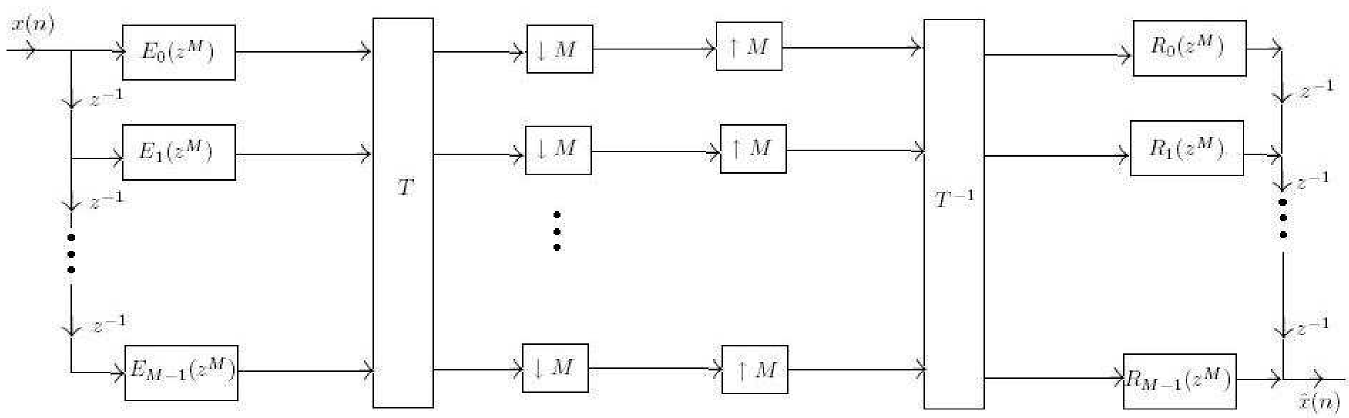

Figure : P-2

a) First as a review, verify that the uniform-shift relations $H_k(z) = H_0(zW^k)$ and $F_k(z) = W^{-k}F_0(zW^k)$ hold, where $W = e^{-\frac{j2\pi}{M}}$.

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b) Express the distortion function T(z) in terms of $E_k(z)$, $0 \leq k \leq M-1$.

c) Show that the AC matrix $\mathbf{H}$(z) is left circulant.

d) Find the determinant of $\mathbf{H}$(z) in terms of $E_k(z)$. Show that this determinant is equal to $cz^{-K}T(z)$ where $c \neq 0$ and K is some integer. Thus, $\mathbf{H}(e^{j\theta})$ is singular if and only if $T(e^{j\theta}) = 0$. Hint: based on the polyphase representation discussed in the lecture, show first that for any M-channel maximally decimated filter bank, the relation between the polyphase matrix $\mathbf{E}$(z) and AC matrix $\mathbf{H}$(z) is

$$\mathbf{H}(z) = \mathbf{W}^H \mathbf{D}(z) \mathbf{E}^T(z^M), \tag{1}$$

where $\mathbf{D}(z) = diag[1, z^{-1}, ..., z^{-M+1}]$ is a diagonal matrix, and $\mathbf{W^H}$ is the transpose conjugate of the $M \times M$ DFT matrix.

e) Suppose $H_0(z) = \sum_{n=0}^{N} h[n]z^{-n}$. Assume h[n] is real with h[n]=h[N-n] (linear phase FIR). This property imposes certain constraints on the polyphase components $E_k(z)$. In particular, for some combinations of N and M, it is possible that some polyphase components $E_L(z)$ is an *odd* order filter with *symmetric* impulse response. This implies $E_L(e^{j\pi}) = 0$. Using part (b) prove that this implies $T(e^{jw}) = 0$ for $w = \frac{\pi}{M}, \frac{3\pi}{M}, \ldots,$ etc. To avoid this situation, the relative values of M and N must be carefully chosen. Explain how. (Hint: You can do it using the notation m_0 as explained below. For M=2, this should reduce to the requirement that N be odd, as seen in the lecture.)

Note on the m_0 notation (as shown in Homework#2): Let $H(z) = \sum_{n=0}^{N} h[n]z^{-n}$ with $h[n] = h[N-n]$. Consider the Type-1 polyphase representation for $H(z)$. The symmetry of $h[n]$ reflects into the coefficients of the polyphase components $E_l(z)$: there exists an integer m_0 (with $0 \leq m_0 \leq M-1$) such that $e_k[n]$ is the image of $e_{m_0-k}[n]$ for $0 \leq k \leq m_0$, and $e_k[n]$ is the image of $e_{M+m_0-k}[n]$ for $m_0+1 \leq k \leq M-1$.

Problem 3 Suppose we have a three-channel alias-free QMF bank with distortion function $T(z) = \frac{z^{-2}}{1-az^{-1}}$. Find closed form expression for the elements of the 3x3 matrix $\mathbf{P}$(z) in terms of a and z.

Problem 4 Consider the M-channel analysis/synthesis system in Figure P-4(a). This reduces to the traditional QMF bank if L=M. If $L < M$, this is called a non-maximally decimated QMF bank. With such a system, elimination of aliasing turns out to be relatively easy.

a) Find an expression for $\hat{X}$(z).

b) Suppose M=4 and L=3. Suppose the analysis bank is a uniform DFT bank, i.e. $H_k(z) = H(zW_4^k), 0 \leq k \leq 3$. Assume that $H_0(z)$ has a response shown in Figure P-4(b). How small would ϵ have to be so that $H_0(z)$ does not overlap with the aliased versions $H_0(zW_3^n), n = 1, 2$? With such ϵ, show typical responses of $F_k(z), 0 \leq k \leq 3$ such that aliasing terms are eliminated. (The trivial choice $F_k(z) = 0$ is obviously forbidden.) What is the distortion transfer function after such elimination of aliasing?

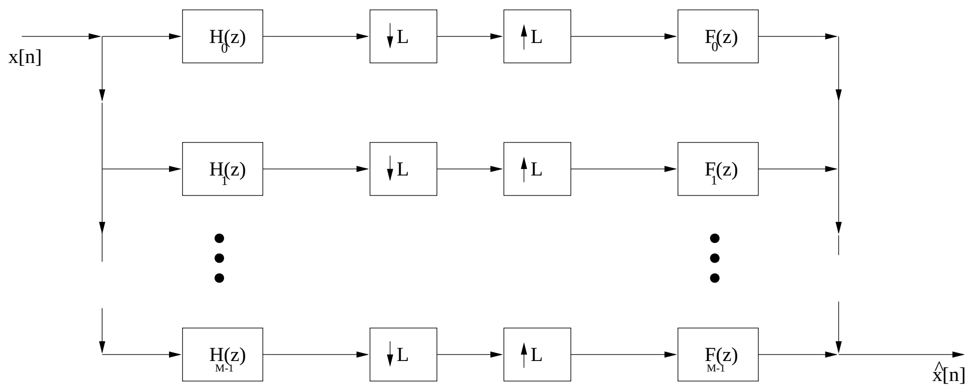

Figure : P-4(a)

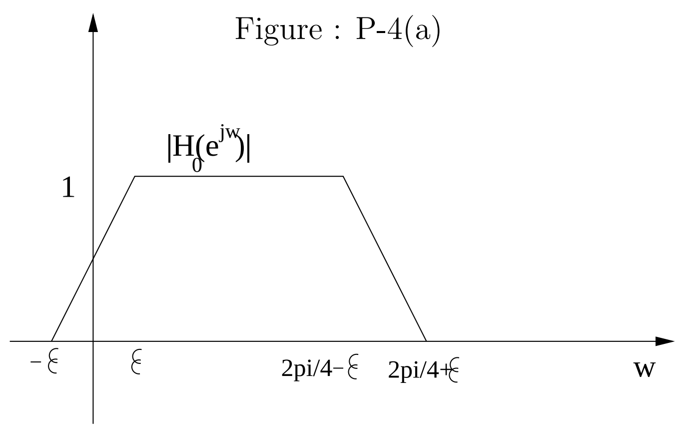

Figure : P-4(b)

Note: In non-maximally decimated systems the total number of samples per unit time at the output of the decimated analysis bank is evidently more than that for x[n]. This is the price paid to obtain the simplicity of alias elimination.