ENEE 630 Homework 9[1]

Take-home Exercise. Solution will be posted on the course website

Problem 1 Find the variance of the unbiased ACF estimator

$$\hat{r}'_{xx}[k] = \frac{1}{N-k} \sum_{n=0}^{N-1-k} x[n]x[n+k] \quad 0 \leq k \leq N-1$$

for real data which is a zero-mean white Gaussian process with variance σ_x^2. What happens as the lag k increases?

Repeat the problem for the biased ACF estimator and explain your results:

$$\hat{r}_{xx}[k] = \frac{1}{N} \sum_{n=0}^{N-1-k} x[n]x[n+k]$$

Hint: First prove that for any real zero-mean Gaussian process the variance of the unbiased ACF estimator is

$$var[\hat{r}'_{xx}[k]] = \frac{1}{N-k} \sum_{j=-(N-1-k)}^{N-1-k} \left(1 - \frac{|\, j \,|}{N-k}\right) (r_{xx}^2[j] + r_{xx}[j+k]r_{xx}[j-k]).$$

Problem 2 Show that the exponential signal

$$x[n] = \sum_{i=1}^{p} A_{c_i} z_i^n$$

may be generated by the recursive difference equation

$$x[n] = -\sum_{k=1}^{p} a[k]x[n-k]$$

for $n \geq p$ under the following conditions:

(1) If the $a[k]'s$ are chosen so that the zeros of the polynomial

$$\mathcal{A}(z) = 1 + \sum_{k=1}^{p} a[k]z^{-k}$$

are $z_1, z_2, ..., z_p$, and

(2) If the initial conditions of the difference equations are chosen as

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[p-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_p \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{p-1} & z_2^{p-1} & \dots & z_p^{p-1} \end{bmatrix} \begin{bmatrix} A_{c_1} \\ A_{c_2} \\ \vdots \\ A_{c_p} \end{bmatrix}$$

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Problem 3 In this problem, a non-causal IIR Wiener filter for noise smoothing is derived. Assume that the real process $y[n] = x[n] + w[n]$ is observed for $-\infty < n < \infty$. $x[n]$ is a zero-mean real WSS random process with PSD $P_{xx}(f)$ and $w[n]$ is a zero-mean real white noise process with variance σ_w^2 which is uncorrelated with $x[n]$. The Wiener estimate of $x[n]$ for $-\infty < n < \infty$ is found as

$$\hat{x}[n] = \sum_{k=-\infty}^{\infty} h[k]y[n-k]$$

where the impulse response $h[n]$ is chosen to minimize the mean square error

$$MSE = \mathcal{E}\{(x[n] - \hat{x}[n])^2\}$$

Show that the impulse response must satisfy

$$r_{yx}[m] = \sum_{k=-\infty}^{\infty} h[k]r_{yy}[m-k] \quad -\infty < m < \infty$$

Use this result to show that the frequency response of the Wiener filter is given by

$$H(f) = \frac{P_{xx}(f)}{P_{xx}(f) + \sigma_w^2}$$

Comment on the use of a Wiener filter for noise reduction in AR spectral estimation.