Material covered: Transforms and properties of the transform function of a discrete-time LTI system.

1. Define $H(z) = \sum_{n=-\infty}^{+\infty} h(n)z^{-n}$.

(1). If $h(n) = a^n u(n)$, what is $H(z)$?

(2). If $h(n) = -a^n u(-n-1)$, what is $H(z)$?

(3). Given $H(z) = \frac{3}{1-0.5z^{-1}} + \frac{1}{1+0.75z^{-1}} + \frac{1}{1-2z^{-1}}$, what are the possible $h(n)$? If $h(n)$ is the impulse response of an LTI system, discuss whether the system is causal or stable.

Solution:

(1) By definition. $H(z) = \sum_{n=0}^{+\infty} a^n z^{-n} = \frac{1}{1-az^{-1}}$ with R.O.C $|z| > |a|$.

(2) By definition. $H(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \frac{1}{1-az^{-1}}$ with R.O.C $|z| < |a|$.

(3) Four possible cases.

i. $|z| < 0.5$, $h(n) = -3(0.5)^n u(-n-1) - (-0.75)^n u(-n-1) - 2^n u(-n-1)$.

ii. $0.5 < |z| < 0.75$, $h(n) = 3(0.5)^n u(n) - (-0.75)^n u(-n-1) - 2^n u(-n-1)$.

iii. $0.75 < |z| < 2$, $h(n) = 3(0.5)^n u(n) + (-0.75)^n u(n) - 2^n u(-n-1)$.

iv. $|z| > 2$, $h(n) = 3(0.5)^n u(n) + (-0.75)^n u(n) + 2^n u(n)$.

Only the fourth case is causal by definition. Only the third case is stable since the unit circle $|z| = 1$ lies inside the R.O.C.

Note:

1. R.O.C is important. For a right-sided sequence as in (1), R.O.C is outside; for a left-sided sequence as in (2), R.O.C is inside.

2. The general condition of a system being stable is that the unit circle ($|z| = 1$) lies within the R.O.C. However, since usually we are interested in the causal system (must be right-sided), its corollary is more well-known: all poles should be inside the unit circle. (Otherwise, if p_0 is a pole outside the unit circle, the R.O.C is $|z| > |p_0| > 1$, which is impossible for the unit circle to lie within.)

2. A causal system can be represented by the block diagram in Fig. R1.2.

(1). What is the system transfer function $H(z)$?

(2). Given $a_0 = 1, a_1 = -0.1, a_2 = -0.9$, and $b_1 = 0.9, b_2 = 0.81$, is the system stable? minimum phase? Sketch its frequency response $|H(e^{j\omega})|$.

(3). Suppose b_1 and b_2 are real numbers, prove that $b_2 < 1$, $b_1 + b_2 > -1$, and $b_1 - b_2 < 1$ is the condition that the system is stable. (Hint: discuss the cases when the two roots are real and when the roots are complex conjugate.)

Solution:

(1) $H(z) = \frac{a_0+a_1z^{-1}+a_2z^{-2}}{1+b_1z^{-1}+b_2z^{-2}}$

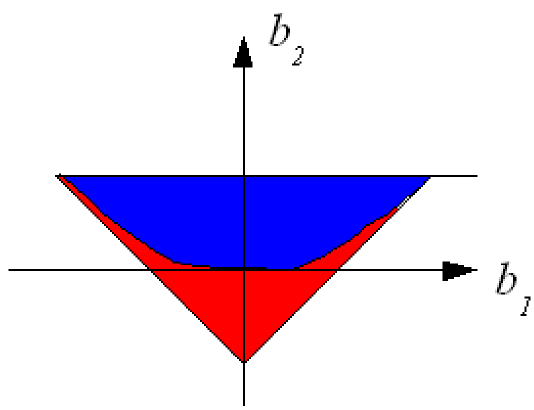
(2) Solving $1 - 0.1z^{-1} - 0.9z^{-2} = 0$ to get the zeros $z_1 = 1$ and $z_2 = -0.9$. Similarly, solving $1 + 0.9z^{-1} + 0.81z^{-2} = 0$ to get the poles $p_{1,2} = 0.9e^{\pm j\frac{2\pi}{3}}$. Stable. Not Minimum phase. Frequency response: 0 at $\omega = 0$, a valley at $\omega = \pi$, and two peaks at $\omega = \pm\frac{2\pi}{3}$ (or equivalently, $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$).

(3) The system is stable iff both poles are inside the unit circle. Two cases:

i. Two roots are complex conjugate when $b_1^2 - 4b_2 < 0$. The poles are $z_{1,2} = \frac{-b_1\pm j\sqrt{4b_2-b_1^2}}{2}$. Let $|z_{1,2}| < 1$, we have $b_2 < 1$. The condition is their intersection, i.e., the blue part in the figure.

ii. Two roots are real when $b_1^2 - 4b_2 \geq 0$. The smaller pole is $z_s = \frac{-b_1-\sqrt{b_1^2-4b_2}}{2}$, whereas the larger one is $z_l = \frac{-b_1+\sqrt{b_1^2-4b_2}}{2}$. Let $z_s > -1$ and $z_l < 1$, we have $b_1 + b_2 > -1$ and $b_1 - b_2 < 1$. The condition is their intersection, i.e., the red part in the figure.

To sum up, the overall condition is the union of case i. and case ii., which is the triangle region in the figure (blue + red).



Note:

1. For a causal system, check its poles to tell whether it is stable.

2. Minimum phase system (when causal), requiring both zeros and poles are inside the unit circle, implies both the original system and its inverse system are stable: the poles of the original system, as well as the poles of the inverse system (and hence the zeros of the original system) must lie inside the unit circle.

3. If $X(z)$ and $Y(z)$ are z-transform of $x(n)$ and $y(n)$, respectively. Prove the following:

(1). The z-transform of $x^*(n)$ is $X^*(z^*)$ (Conjugate).

(2). The z-transform of $x(-n)$ is $X(1/z)$ (Time reversal).

(3). If $R_{xy}(k) = \sum_{n=-\infty}^{+\infty} x(n)y^*(n-k)$, its z-transform $S_{xy}(z) = X(z)Y^*(\frac{1}{z^*})$ (Cross correlation).

Solution: By definition.

(1) $\sum_{n=-\infty}^{+\infty} x^*(n)z^{-n} = \sum_{n=-\infty}^{+\infty} x^*(n)((z^*)^{-n})^* = (\sum_{n=-\infty}^{+\infty} x(n)((z^*)^{-n}))^* = X^*(z^*)$.

(2) $\sum_{n=-\infty}^{+\infty} x(-n)z^{-n} = \sum_{m=-\infty}^{+\infty} x(m)z^{m} = \sum_{m=-\infty}^{+\infty} x(m)(z^{-1})^{-m} = X(1/z)$.

(3) $\sum_{k=-\infty}^{+\infty} R_{xy}(k)z^{-k} = \sum_{k=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} x(n)y^*(n-k)z^{-k} = \sum_{n=-\infty}^{+\infty}\sum_{t=-\infty}^{+\infty} x(n)y^*(t)z^{t-n} = X(z)Y^*(\frac{1}{z^*})$.

Note:

1. $z = \left(z^{-1}\right)^{-1}$, $z = (z^*)^*$.
2. Replacing the variables and/or interchanging the order of summations may help.

4. See the block diagram shown in Fig. R1.4.

(1). For the two-input two-output sub-block G, what are the impulse responses $h_{ij}(n)$ (i.e., from input x_j to output y_i, $i,j = 1,2$)?

(2). For the whole system, what is the impulse response?

Solution:

(1) Let vector $\boldsymbol{X}(z) = [X_1(z), X_2(z)]^T$, and $\boldsymbol{Y}(z) = [Y_1(z), Y_2(z)]^T$, then $\boldsymbol{Y}(z) = \boldsymbol{H}(z)\boldsymbol{X}(z)$, where

$$\boldsymbol{H}(z) = \begin{bmatrix} H_{11}(z) & H_{12}(z) \\ H_{21}(z) & H_{22}(z) \end{bmatrix}.$$

Here $H_{ij}(z)$ is from input j to output i.

Use an intermediate variable $\boldsymbol{U}(z)$, $\boldsymbol{U}(z) = \boldsymbol{H}_1(z)\boldsymbol{X}(z)$ and $\boldsymbol{Y}(z) = \boldsymbol{H}_2(z)\boldsymbol{U}(z)$, where

$$\boldsymbol{H}_1(z) = \begin{bmatrix} 1 & 0 \\ z^{-1} & 3 \end{bmatrix}$$

and

$$\boldsymbol{H}_2(z) = \begin{bmatrix} z^{-2} & 1 \\ 1 & z^{-1} \end{bmatrix}.$$

Therefore,

$$\boldsymbol{H}(z) = \boldsymbol{H}_2(z)\boldsymbol{H}_1(z) = \begin{bmatrix} z^{-1}+z^{-2} & 3 \\ 1+z^{-2} & 3z^{-1} \end{bmatrix}.$$

(2) Similarly,

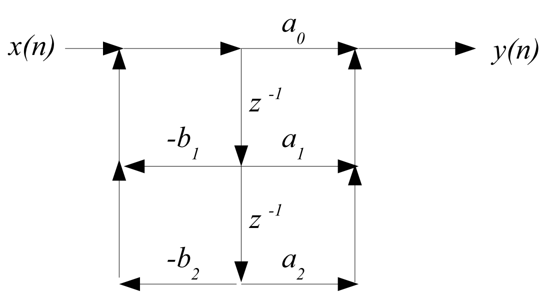
$$\left[1-z^{-1}, 1\right] \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \boldsymbol{G}(z) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} = 2 + 8z^{-1} + 2z^{-2} - 2z^{-3}.$$

Note:

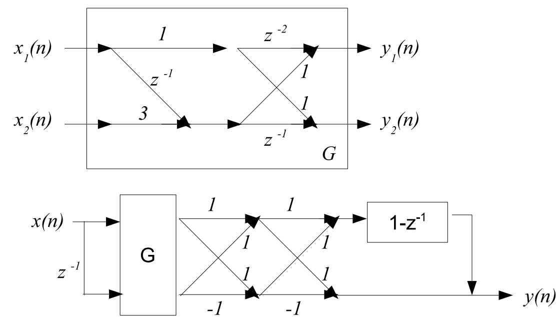
1. An alternative way to calculate $H_{ij}(z)$ from input j to output i is letting all inputs except j be zero, then get the response. It's okay if you do it this way, although familiarizing yourself with matrix representation may be helpful.

2. When we define input/output in column vectors as we did in the above solution, the rows of $\boldsymbol{H}$ correspond to the output, whereas the columns correspond to the input, i.e., $H_{ij}(z)$, the entry in the i-th row and j-th column, is from input j to output i.

3. For matrices, $\boldsymbol{AB} \neq \boldsymbol{BA}$ in general. Therefore, you should be careful about the order of matrices.



(a) Figure.R1.2



(b) Figure.R1.4