1. Consider the structures shown in Fig. RI.1, with input transforms and filter responses as indicated. Sketch the quantities $Y_0(e^{j\omega})$ and $Y_1(e^{j\omega})$.

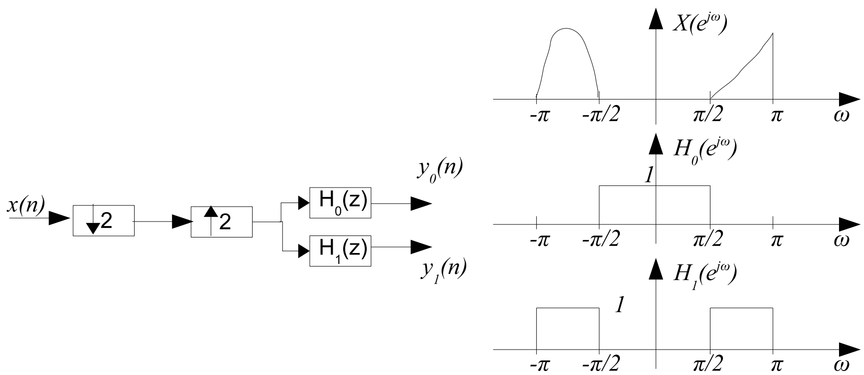
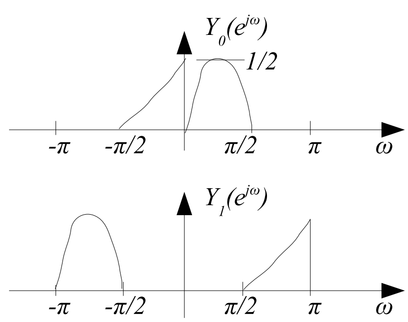


Figure RI.1:

Solution:



Comment: For a down-sampled signal, it's still possible to recover the original signal (not necessarily low-pass) using filters and multirate building blocks as long as there is no aliasing (H_1 in this problem).

2. For each case shown in the Fig. RI.2, prove or disprove whether the left system is equivalent to the right system? Assume M, L, K are all integers larger than 1.

Solution: Assume the input, output, intermediate signals are $x(n), y(n)$, and $u(n)$, respectively.

(a). (FD) $U(z) = X(z^M)$, $Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} U(z^{1/M}W_M^k) = \frac{1}{M}\sum_{k=0}^{M-1} X(zW_M^{kM}) = X(z)$.
(TD) $u(n) = x(\frac{n}{M})$ if n is a multiple of M, and $u(n) = 0$ otherwise. Then, $y_2(n) = u(Mn) = x(n)$.

(b). (FD) $U(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^k)$, $Y_2(z) = U(z^M) = \frac{1}{M}\sum_{k=0}^{M-1} X(zW_M^k)$.
(TD) $u(n) = x(Mn)$, $y_2(n) = u(\frac{n}{M}) = x(n)$ only if n is a multiple of M, i.e., $y_2(n) = \begin{cases} x(n) & n \text{ is multiple of } M \\ 0 & \text{otherwise.} \end{cases}$

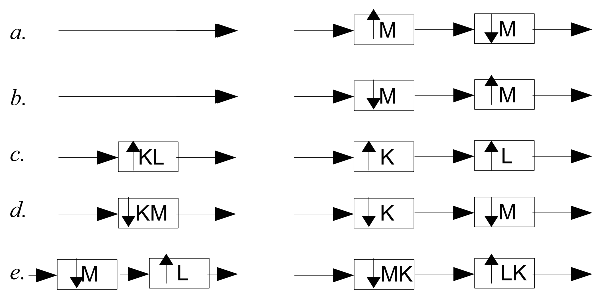


Figure RI.2:

(c). (FD) $U(z) = X(z^K)$, $Y_2(z) = U(z^L) = X(z^{KL})$.
(TD) $u(n) = x(\frac{n}{K})$ only if n is a multiple of K, and $y_2(n) = u(\frac{n}{L})$ only if n is a multiple of L. Zero otherwise. Hence, $y_2(n) = x(\frac{n}{KL})$ only when n is a multiple of KL.

(d) (FD) $U(z) = \frac{1}{K}\sum_{k=0}^{K-1} X(z^{1/K} W_K^k)$, $Y_2(z) = \frac{1}{M}\sum_{m=0}^{M-1} U(z^{1/M} W_M^m)$
$= \frac{1}{MK}\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} X(z^{1/KM} e^{-j\frac{2\pi m}{MK}} e^{-j\frac{2\pi kM}{MK}}) = \frac{1}{MK}\sum_{m=0}^{M-1}\sum_{k=0}^{K-1} X(z^{1/KM} W_{MK}^{kM+m})$
$= \frac{1}{MK}\sum_{t=0}^{MK-1} X(z^{1/KM} W_{MK}^t)$
(TD) $u(n) = x(Kn)$, $y_2(n) = u(Mn) = x(KMn)$.

(e) (FD) Left: $Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{L/M} W_M^k)$; Right: $Y_2(z) = \frac{1}{MK}\sum_{k=0}^{MK-1} X(z^{L/M} W_{MK}^k)$
(TD) Left: $y_2(n) = x(n\frac{M}{L})$ only when n is a multiple of L; right: $y_2(n) = x(n\frac{M}{L})$ only when n is a multiple of KL.

Note: To prove/disprove the equivalence, you can prove either in time domain or in freq domain, no need to do both. You are recommended to try all on your own (except (d) freq domain) to familiar yourself with the derivation.

3. Simplify the following systems in Fig. I.3.

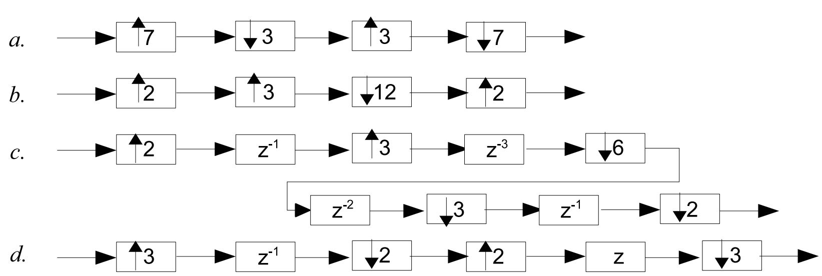
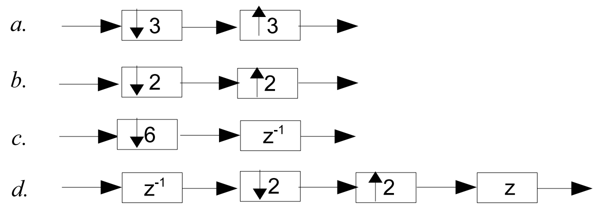


Figure RI.3:

Solution: The simplified systems are as follows
Note: The basic relationships from the previous problem and Nobel identities are applied. For part (d), the trick is $z = z^3 z^{-2}$ and $z^{-1} = z^{-3} z^2$.



4. In this problem, the term 'polyphase components' stands for the Type 1 components with $M = 2$.

(a) Let $H(z)$ represent an FIR filter of length 10 with impulse response coefficients $h(n) = (1/2)^n$ for $0 \leq n \leq 9$ and zero otherwise. Find the polyphase components $E_0(z)$ and $E_1(z)$.

(b) Let $H(z)$ be IIR with $h(n) = (1/2)^n u(n) + (1/3)^n u(n-3)$. Find the polyphase components $E_0(z)$ and $E_1(z)$. Give simplified, closed form expressions. (Hint: $\frac{1}{1+x} = \frac{1-x}{1-x^2}$)

(c) Let $H(z) = 1/(1 - 2R\cos\theta z^{-1} + R^2 z^{-2})$, with $R > 0$ and θ real. This is a system with a pair of complex conjugate poles at $Re^{\pm j\theta}$. Find the polyphase components $E_0(z)$ and $E_1(z)$.

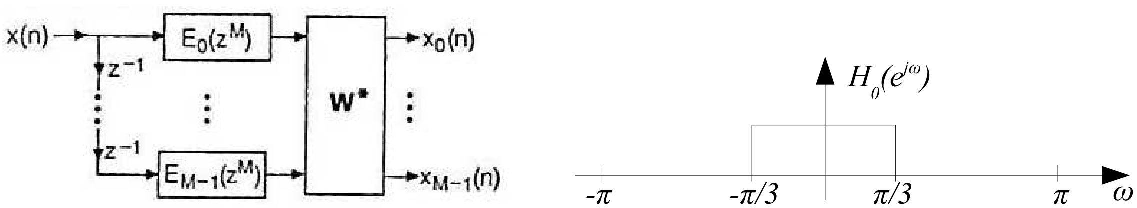
Solution:

(a). $E_0(z) = 1 + (1/2)^2 z^{-1} + (1/2)^4 z^{-2} + (1/2)^6 z^{-3} + (1/2)^8 z^{-4}$, $E_1(z) = 1/2 + (1/2)^3 z^{-1} + (1/2)^5 z^{-2} + (1/2)^7 z^{-3} + (1/2)^9 z^{-4}$.

(b). $H(z) = \frac{1}{1-(1/2)z^{-1}} + \frac{(1/3)^3 z^{-3}}{1-(1/3)z^{-1}} = \frac{1+(1/2)z^{-1}}{1-(1/4)z^{-2}} + \frac{(1/3)^3 z^{-3} + (1/3)^4 z^{-4}}{1-(1/9)z^{-2}}$. Hence, $E_0(z) = \frac{1}{1-(1/4)z^{-1}} + \frac{(1/3)^4 z^{-2}}{1-(1/9)z^{-1}}$, and $E_1(z) = \frac{1/2}{1-(1/4)z^{-1}} + \frac{(1/3)^3 z^{-1}}{1-(1/9)z^{-1}}$.

(c). $H(z) = \frac{1}{1-2R\cos\theta z^{-1}+R^2 z^{-2}} = \frac{(1+2R\cos\theta z^{-1}+R^2 z^{-2})}{(1-2R\cos\theta z^{-1}+R^2 z^{-2})(1+2R\cos\theta z^{-1}+R^2 z^{-2})}$. Hence, $E_0(z) = \frac{1+R^2 z^{-1}}{(1+R^2 z^{-1})^2 - 4R^2\cos^2\theta z^{-1}}$, and $E_1(z) = \frac{2R\cos\theta}{(1+R^2 z^{-1})^2 - 4R^2\cos^2\theta z^{-1}}$

5. A uniform DFT analysis bank (Type 1) is shown in Fig. RI.5(a), where $\boldsymbol{W}^*$ is the $M \times M$ IDFT matrix, i.e., the (m, n)-th entry is W_M^{-mn} with indices m, n starting from 0. The transfer function from input port $x(n)$ to output port $x_k(n)$ is denoted by $H_k(z)$. Answer the following questions.



(a) (b)

Figure RI.5:

(a). Prove $H_k(z) = H_0(zW_M^k)$ for $0 \leq k \leq M-1$. Given $H_0(e^{j\omega})$ in Fig. RI.5(b), sketch $H_1(e^{j\omega})$ when $M = 2$, and $H_1(e^{j\omega})$, $H_2(e^{j\omega})$ when $M = 3$.

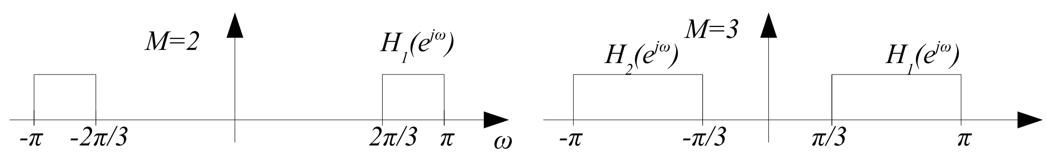
(b). $M = 4$. Assume $E_0(z) = 1 + z^{-1}$, $E_1(z) = 1 + 2z^{-1}$, $E_2(z) = 2 + z^{-1}$, and $E_3(z) = 0.5 + z^{-1}$.

Find numerical values of these filter coefficients for $H_k(z)$, $0 \le k \le 3$.

(c). $M = 2$. Let $H_0(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$, and let $H_1(z) = H_0(-z)$. Draw an implementation for the pair $[H_0(z), H_1(z)]$ in the form of a uniform DFT analysis bank, explicitly showing the polyphase components, the 2×2 IDFT box, and other relevant details.

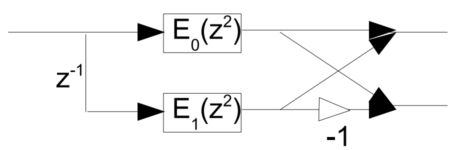
Solution:

(a). $H_0(z) = \sum_{m=0}^{M-1} z^{-m} E_m(z^M)$, $H_k(z) = \sum_{m=0}^{M-1} z^{-m} E_m(z^M) W_M^{-mk} = \sum_{m=0}^{M-1} (zW_M^k)^{-m} E_m((zW_M^k)^M) = H_0(zW_M^k)$. $H_k(e^{j\omega}) = H_0(e^{j(\omega - 2\pi k/M)})$.



(b). $H_0(z) = 1 + z^{-1} + 2z^{-2} + 0.5z^{-3} + z^{-4} + 2z^{-5} + z^{-6} + z^{-7}$. $H_1(z) = 1 + jz^{-1} - 2z^{-2} - 0.5jz^{-3} + z^{-4} + 2jz^{-5} - z^{-6} - jz^{-7}$. $H_2(z) = 1 - z^{-1} + 2z^{-2} - 0.5z^{-3} + z^{-4} - 2z^{-5} + z^{-6} - z^{-7}$. $H_3(z) = 1 - jz^{-1} - 2z^{-2} + 0.5jz^{-3} + z^{-4} - 2jz^{-5} - z^{-6} + jz^{-7}$. ($j = \sqrt{-1}$).

(c) Note that $W_2^1 = -1$, then they can be implemented together using one DFT analysis bank. $E_0(z) = 1 + 4z^{-1} + z^{-2}$, $E_1(z) = 2 + 2z^{-1}$.



Note:

1. $W_M^1 = e^{-j\frac{2\pi}{M}}$, e.g., $W_2^1 = -1$, $W_4^1 = -j$.

2. $\boldsymbol{W}^*$ (with entry W_M^{-ij}) is called the IDFT matrix, whereas $\boldsymbol{W}$ (with entry W_M^{ij}) is called the DFT matrix. Note that the index starts from 0 and goes up to $M-1$. One important property is that $\boldsymbol{W}^*\boldsymbol{W} = M\boldsymbol{I}$. For instance, when $M = 2$, $\boldsymbol{W}^*$ and $\boldsymbol{W}$ happen to be the same

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and hence using the property,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which you will encounter very often when studying the QMF bank.

3. Type 2 DFT bank (synthesis bank) is found in Problem #1 in Homework #2. Similar relationship ("shifted version") can be found between transfer functions.

4. Given the DFT bank structure, we can expect transfer functions have relationship (a); on the other hand, given the transfer functions have the relationship, we can implement it by the DFT structure. Part (b) and part (c) of this problem show the two aspects.

5. For $H_0(z)$, since all weights are 1, it is just the type 1 polyphase representation, and hence $H_0(z) = \sum_{m=0}^{M-1} z^{-m} E_m(z^M)$.

7. Prove or disprove: the systems are linear time-invariant (LTI).

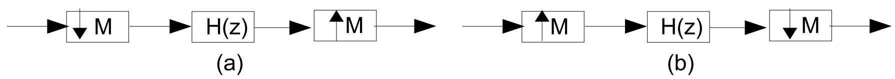
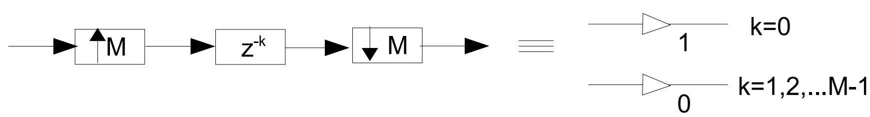


Figure RI.7:

Solution:

(a) NO. $U(z) = H(z)\frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^k)$, $Y(z) = U(z^M) = \frac{H(z^M)}{M}\sum_{k=0}^{M-1} X(zW_M^k)$

(b) YES. $U(z) = H(z)X(z^M)$, and $Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} U(z^{1/M}W_M^k) = \frac{1}{M}\sum_{k=0}^{M-1} H(z^{1/M}W_M^k)X(z) = \left[\frac{1}{M}\sum_{k=0}^{M-1} H(z^{1/M}W_M^k)\right] X(z)$. Actually, the effective transfer function is just the polyphase component $E_0(z)$. An alternative way is using the equivalence



Note: A system is LTI if and only if it can be represented by $H(z)$, i.e., $Y(z) = H(z)X(z)$.

8. Prove that the block diagram shown in Fig. R4.2(a) is a perfect reconstruction system. Based on this, answer the following questions (No knowledge of QMF bank is needed though. We will briefly revisit this problem in next recitation):

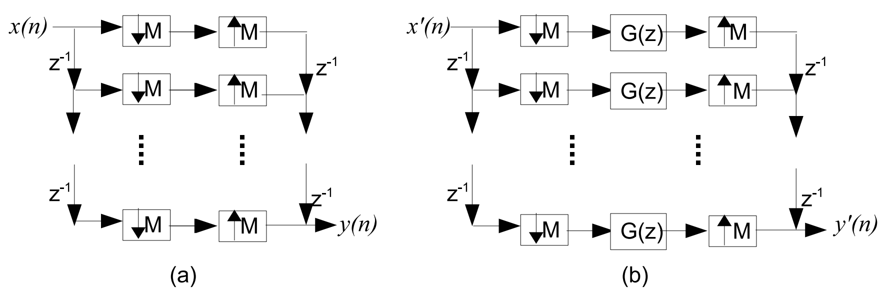


Figure RI.8:

(a) What is the transfer function $H'(z)$ for the block diagram in Fig. R4.2(b)?

(b) Suppose that we are to design a system with the desired system function $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 6z^{-5} + 7z^{-6}$ in a very high-speed environment. If the speed of the devices used to implement the system is three times slower than the required operating speed, please devise a scheme that can implement $H(z)$ under the constraint of the device. (Hint: implement $z^{-2}H(z)$)

Solution:

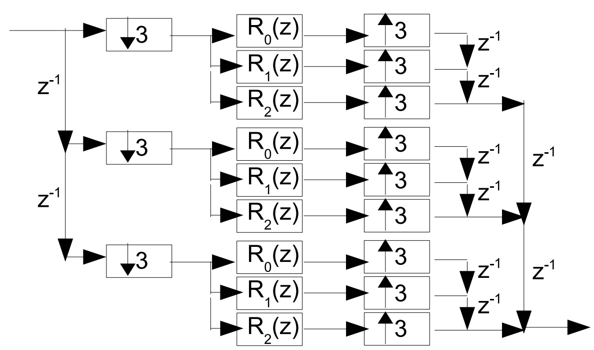
(FD) $U_0(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^k), V_0(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(zW_M^k); U_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} z^{-1/M}W_M^{-k}X(z^{1/M}W_M^k)$, $V_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} z^{-1}W_M^{-k}X(zW_M^k); \cdots$ In general, $U_m(z) = \frac{1}{M}\sum_{k=0}^{M-1} z^{-m/M}W_M^{-mk}X(z^{1/M}W_M^k), V_m(z) = \frac{1}{M}\sum_{k=0}^{M-1} z^{-m}W_M^{-mk}X(zW_M^k)$. As a result, $Y(z) = \sum_{m=0}^{M-1} z^{-(M-1-m)}V_m(z)$ $= z^{-(M-1)}\frac{1}{M}\sum_{m=0}^{M-1}\sum_{k=0}^{M-1} W_M^{-mk}X(zW_M^k)$. Use the fact that $\sum_{m=0}^{M-1} W_M^{-mk} = M$ when $k = 0$ and $\sum_{m=0}^{M-1} W_M^{-mk} = 0$ when $k = 1, 2\ldots, M-1$, we get $H(z) = z^{-(M-1)}$.

or

(TD) $u_0(n) = x(Mn), v_0(n) = u_0(n/M) = x(n)$ only if $\mod(n, M) = 0$. Let $w_0(n) = v_0(n - M + 1)$, then $w_0(n) = x_0(n - M + 1)$ only if $\mod(n, M) = M - 1$ and 0 otherwise. $u_1(n) = x(nM - 1), v_1(n) = u_1(n/M) = x(n-1)$ only if $\mod(n, M) = 0$. Let $w_1(n) = v_1(n - M + 2)$, then $w_1(n) = x(n - M + 1)$ only if $\mod(n, M) = M - 2$ and 0 otherwise. In general, $w_m(n) = x(n - M + 1)$ only if $\mod(n, M) = M - m - 1$. $y(n) = \sum_{m=0}^{M-1} w_m(n) = x(n - M + 1)$.

(a). $z^{-(M-1)}G(z^M)$.

(b). $R_0(z) = 3 + 6z^{-1}, R_1(z) = 2 + 5z^{-1}, R_2(z) = 1 + 4z^{-1} + 7z^{-2}$.



9. Consider the QMF bank in Fig. R4.3, where $H_0(z) = 1 + 2z^{-1} + 3z^{-2} + z^{-3}$, and $H_1(z) = H_0(-z)$.

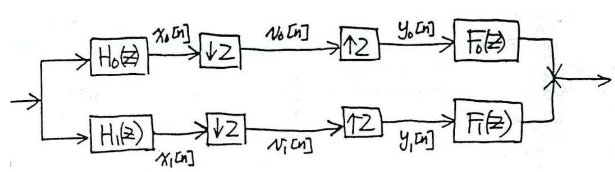


Figure RI.9:

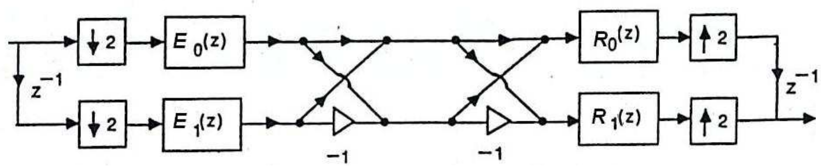
(a) Design an FIR synthesis bank $F_0(z), F_1(z)$ such that the system is alias-free.

(b) Design an IIR synthesis bank $F_0(z), F_1(z)$ such that the system is perfect reconstruction. Is this bank stable? (By default, we assume it is causal)

(c) Design a stable IIR synthesis bank $F_0(z), F_1(z)$ such that the system is alias-free and amplitude-distortion-free. (Hint: $\frac{a^* + z^{-1}}{1 + az^{-1}}$ is an all-pass filter.)

Solution:

$E_0(z) = 1 + 3z^{-1}$, $E_1(z) = 2 + z^{-1}$.



$F_0(z) = z^{-1}R_0(z^2) + R_1(z^2)$, $F_1(z) = -F_0(-z) = z^{-1}R_0(z^2) - R_1(z^2)$.

(a) $R_0(z) = E_1(z)$, $R_1(z) = E_0(z)$. $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$.

(b) $R_0(z) = 1/E_0(z)$, $R_1(z) = 1/E_1(z)$. $T(z) = 2z^{-1}$. However, check that $R_0(z)$ is not stable.

(c) Let $\widetilde{E}_0(z) = 3 + z^{-1}$, then $\frac{E_0(z)}{\widetilde{E}_0(z)}$ is an all-pass filter. $R_0(z) = \frac{1}{\widetilde{E}_0(z)}$, $R_1(z) = \frac{E_0(z)}{\widetilde{E}_0(z)E_1(z)}$. $T(z) = 2z^{-1}\frac{E_0(z^2)}{\widetilde{E}_0(z^2)}$.

Note:

1. When designing a QMF bank, we usually use the polyphase decomposition as in this problem. In this problem, since the IDFT and DFT matrix cancel out each other, we get a structure similar to Fig. R4.2(b) with each branch being $E_m(z)R_m(z)$. To make it alias-free, we require all branches should be the same, i.e., $E_0(z)R_0(z) = E_1(z)R_1(z)$. If we further desire a system that is amplitude-distortion-free or P.R., then not only all branches should be the same, but also $|E_m(z)R_m(z)| = const$ (for amp-distortion-free) or $E_m(z)R_m(z) = cz^{-n_0}$ (for P.R.). We will discuss more in the next recitation.

2. In general, if $A(z) = a_0 + a_1z^{-1} + a_2z^{-2} + \ldots + a_nz^{-n}$, then define $\widetilde{A}(z) = a_n^* + a_{n-1}^*z^{-1} + a_{n-2}^*z^{-2} + \ldots + a_0^*z^{-n}$ (reverse the order and take conjugate), and $\frac{\widetilde{A}(z)}{A(z)}$ is an all-pass filter. For example, $\frac{a+z^{-1}}{1+az^{-1}}$ is the 1-order real-coefficient all-pass filter.