1. Assume that $v(n)$ is a real-valued zero-mean white Gaussian noise with $\sigma_v^2 = 1$, $x(n)$ and $y(n)$ are generated by the equations

$$x(n) = 0.5x(n-1) + v(n),$$

$$y(n) = x(n-1) + x(n).$$

(a) Find the power spectrum of sequence $x(n)$, and its power.

(b) Find the power spectrum of sequence $y(n)$, and its power.

(c) Calculate $r_y(k)$ for $k = 0, 1, 2, 3$.

Assume now we don't know the real model of the signal, and we want to estimate its power spectrum from $r_y(k)$ obtained in part (c). Estimate power spectrum using the following methods:

(d) ARMA(1,1) spectral estimation.

(e) AR(2) spectral estimation.

(f) Maximum entropy spectral estimation with order 2.

(g) Minimum variance spectral estimation with order 1.

Solution:

(a) $x(n)$ can be modeled as AR(1). Hence,

$$P_x(\omega) = \frac{\sigma_v^2}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})} = \frac{1}{1.25 - \cos\omega}.$$

For AR(1) process, we know that $r_x(k) = (0.5)^{|k|} r_x(0)$ and $r_x(0) = 4/3$.

(b) $y(n)$ can be modeled as ARMA(1,1).

$$P_x(\omega) = \frac{(1+e^{-j\omega})(1+e^{j\omega})}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})} = \frac{2+2\cos\omega}{1.25 - \cos\omega}.$$

$r_y(0) = 2r_x(0) + 2r_x(1) = 4.$

(c) $r_y(0) = 4$. $r_y(1) = 2r_x(1) + r_x(0) + r_x(2) = 3$. $r_y(2) = 3/2$. $r_y(3) = 3/4$.

(d) Since the assumed model perfectly matches the real model, the estimated spectrum is exactly the true spectrum

$$P_{ARMA}(\omega) = \frac{(1+e^{-j\omega})(1+e^{j\omega})}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})} = \frac{2+2\cos\omega}{1.25 - \cos\omega}.$$

(e) Use the Yule-Walker equation,

$$\begin{bmatrix} r(0) \; r(1) \\ r(1) \; r(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} r(1) \\ r(2) \end{bmatrix}.$$

$a_1 = -15/14, a_2 = 3/7$. $r(0) + a_1 r(1) + a_2 r(2) = \sigma^2$, we have $\sigma^2 = 10/7$.

$$P_{AR}(\omega) = \frac{10/7}{(1 - 15/14e^{-j\omega} + 3/7e^{-j2\omega})(1 - 15/14e^{j\omega} + 3/7e^{j2\omega})}.$$

(f) Maximum entropy is equivalent to AR for Gaussian processes, and hence the result is the same with (e).

(g)

$$\boldsymbol{R}^{-1} = \frac{1}{7}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}.$$

$$P_{MVSE}(\omega) = \frac{p+1}{\boldsymbol{e}^H \boldsymbol{R}^{-1} \boldsymbol{e}} = \frac{7}{4 - 3\cos\omega}$$

2. Show that the periodogram spectrum estimator will result in biased results if an N-point rectangular window is applied, i.e., $P_{PER}(\omega) = \frac{1}{N}|\sum_{n=0}^{N-1} x(n)e^{-j\omega n}|^2$ is biased.

Solution:

$E[P_{PER}(\omega)] = E[\frac{1}{N}|\sum_{n=0}^{N-1} x(n)e^{-j\omega n}|^2] = \frac{1}{N}E[\sum_{n=0}^{N-1} x(n)e^{-j\omega n} \sum_{m=0}^{N-1} x^*(m)e^{j\omega m}]$ $= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E[x(n)x^*(m)]e^{-j\omega(n-m)} = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} r(n-m)e^{-j\omega(n-m)} = \frac{1}{N}\sum_{l=-(N-1)}^{N-1}(N-|l|)r(l)e^{-j\omega l} = \sum_{l=-(N-1)}^{N-1}(1-|l|/N)r(l)e^{-j\omega l}$

Note that the true spectrum is the Fourier transform of $\{r(l)\}$, i.e., $P(w) = \sum_l r(l)e^{-j\omega l}$. As a result, $E[P_{PER}(\omega)] = P(w) * w_T(\omega)$ is a "smeared" version of the true spectrum, where $w_T(\omega)$ is the Fourier transform of a triangle waveform (and hence has the form of $\text{sinc}(\cdot)^2$).

3. Consider a wide-sense stationary process consisting of p distinct complex sinusoids in white noise with variance σ^2, i.e.

$$x(n) = \left[\sum_{i=1}^{p} A_i e^{-j(n\omega_i + \phi_i)}\right] + v(n)$$

where A_i and ϕ_i are uncorrelated, and ϕ_i is a uniformly distributed random variable in $[0, 2\pi)$.

(a) Find the autocorrelation function $r(k) = E[x(n)x(n-k)]$.

(b) Find the $(p+1) \times (p+1)$ correlation matrix R.

Solution:

(a) $r(k) = E(x(n)x^*(n-k)) = \sum_{i=1}^{p} P_i e^{-j\omega_i k} + \sigma^2\delta(k)$, since many cross terms are uncorrelated.

(b) Denote the column vector $\boldsymbol{u}_i = (1, e^{j\omega_i}, \ldots, e^{j\omega_i p})^T$. $\boldsymbol{R} = \sum_{i=1}^{p} P_i \boldsymbol{u}_i \boldsymbol{u}_i^H + \sigma^2 \boldsymbol{I} = \boldsymbol{S}\boldsymbol{P}\boldsymbol{S}^H + \sigma^2\boldsymbol{I}$, where $\boldsymbol{P} = \text{diag}(P_1, P_2, \ldots, P_p)$, and $\boldsymbol{S} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p]$ is a $(p+1) \times p$ matrix.

4. Consider a random process

$$x(n) = A\exp[j(n\omega_0 + \phi)] + \alpha_0 v[n] + \alpha_1 v[n-1],$$

where $\{v[n]\}$ is a white noise process with zero mean and variance σ_v^2. The phase ϕ is uniformly distributed over $[0, 2\pi)$ and uncorrelated with $v[n]$; and A, ω_0, α_0, and α_1 are real-valued constants.

(a) Find the autocorrelation function for $\{x[n]\}$ in terms of $A, \omega_0, \alpha_0, \alpha_1$, and σ_v^2. Your solution should provide all the necessary steps and justifications.

(b) Consider the process in $\{x[n]\}$ for the case of $\alpha_0 = 1$ and $\alpha_1 = 0$. First, determine the eigen values of an $M \times M$ correlation matrix of the $\{x[n]\}$ process. Next. suppose we have observed N samples, $x[0], x[1], ..., x[N-1]$. Use equation, diagram, and concise words to describe the average periodogram method for estimating method for estimating the power spectrum density of the $\{x[n]\}$ process.

Solution:

(a) Let us denote $x[n] = s[n] + u[n]$, where and $s[n] = Ae^{j\phi}e^{j\omega_0 n}$ $u[n] \triangleq a_0 v[n] + a_1 v[n-1]$ are zero mean and uncorrelated. Therefore, autocorrelation function $r_x(k)$ can be written as,

$$r_x(k) = r_s(k) + r_u(k),$$

$r_s k$ can be calculated as following,

$$\begin{aligned} r_s(k) &= E[s[n]s*[n-k]] \\ &= E[Ae^{j\phi}e^{j\omega_0 n}.Ae^{-j\phi}e^{-j\omega_0(n-k)}] \\ &= E[A^2 e^{j\omega_0 k}] \\ &= A^2 e^{j\omega_0 k} \end{aligned}$$

$u[n]$ is a 2^{nd} order MA process. So, it's autocorrelation function $r_u(k)$ can be calculated as following,

$$\begin{aligned} r_u(k) &= E[u[n]u*[n-k]] \\ &= E[(a_0 v[n] + a_1 v[n-1])(a_0 v[n-k] + a_1 v[n-k-1])^*] \\ &= (a_0^2 + a_1^2)\sigma_v^2 \delta[k] + (a_0 a_1 \sigma_v^2)(\delta[k-]) + \delta[k+1]) \end{aligned}$$

$$\Rightarrow r_x k = \begin{cases} A^2 + (a_0^2 + a_1^2)\sigma_v^2, & \text{k=0;} \\ A^2 e^{j\omega+0} + a_0 a_1 \sigma_v^2, & |k| = 1; \\ A^2 e^{j\omega_0 k}, & |k| > 1. \end{cases}$$

(b) when $a_0 = 1$, $a_1 = 0$, $r_x(k) = A^2 e^{j\omega_0 k} + \sigma_v^2 \delta[k]$. The autocorrelation matrix can be written as following,

$$\mathbf{R_x} = \mathbf{R_S} + \sigma_v^2 \mathbf{I},$$

$$\text{where } \mathbf{R_S} = A^2 \begin{pmatrix} 1 & e^{j\omega_0} & \dots & e^{-j(M-1)\omega_0} \\ e^{-j\omega_0} & 1 & \dots & \\ \vdots & \ddots & & \\ \vdots & & \ddots & \end{pmatrix} = A^2 \underline{\mathbf{e}}\underline{\mathbf{e}}^H$$

$$\text{where } \underline{\mathbf{e}} = \begin{pmatrix} 1 \\ e^{-j\omega_0} \\ \vdots \\ e^{-j\omega_0(M-1)} \end{pmatrix}$$

Note that, $\mathbf{R_S}$ is Hermitian non-negative definite matrix of rank 1 (all rows can be represented as a scalar multiplied by first row). Therefore, $\mathbf{R_S}$ has one eigenvector corresponding to a positive eigenvalue and $M-1$ eigenvectors to zero eigenvalue. Let us denote the positive eigenvalue as λ_1 and corresponding eigenvector as $\underline{\mathbf{v_1}}$,

$$\mathbf{R_S}\underline{\mathbf{v_1}} = \lambda_1 \underline{\mathbf{v_1}}$$

Note that following equation also holds true,

$$\mathbf{R_S}\underline{\mathbf{e}} = A^2 \underline{\mathbf{e}}\underline{\mathbf{e}}^H \underline{\mathbf{e}} = MA^2 \underline{\mathbf{e}}$$

Therefore, $\underline{\mathbf{e}}$ is the eigenvector $\underline{\mathbf{v_1}}$ and the eigenvalue $\lambda_1 = MA^2$. If $\underline{\mathbf{u}}$ is an eigenvector of $\mathbf{R_x}$, it is also an eigenvector of $\mathbf{R_s}$ and the corresponding eigenvalues differ by σ_v^2.

$$\begin{aligned} \mathbf{R_x}\underline{\mathbf{u}} &= \lambda_x \underline{\mathbf{u}} \\ \mathbf{R_x}\underline{\mathbf{u}} &= (\mathbf{R_s} + \sigma_v^2 \mathbf{I})\underline{\mathbf{u}} \\ &= \mathbf{R_s}\underline{\mathbf{u}} + \sigma_v^2 \underline{\mathbf{u}} \\ \Rightarrow \mathbf{R_s}\underline{\mathbf{u}} &= (\lambda_x - \sigma_v^2)\underline{\mathbf{u}} \end{aligned}$$

Hence, the eigenvalues of $\mathbf{R_x}$ are $MA^2 + \sigma_v^2$ and σ_v^2.

See class notes for average periodogram.

5. Assume the signal $x(n) = acos(\omega n + \phi) + v(n)$, where a is an unknown constant, $v(n)$ is a white Gaussian noise independent of the sinusoid. Suppose we know the autocorrelation coefficients $r(0) = 3$, $r(1) = \sqrt{2}$, and $r(2) = 0$, determine the frequency of the sinusoid ω and the noise power σ_v^2.

Solution:

The cosine wave is two exponential signals with frequencies $\pm\omega$. We have to use 3×3 correlation matrix,

$$\boldsymbol{R} = \begin{bmatrix} 3 & \sqrt{2} & 0 \\ \sqrt{2} & 3 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{bmatrix}.$$

The eigenvalues are $1, 3, 5$; the eigenvector corresponding to the minimum eigenvalue is $(1, -\sqrt{2}, 1)^T$. According to the MUSIC/Pisorenko algorithm, $\sigma_v^2 = 1$, $1 - \sqrt{2}e^{j\omega} + e^{j2\omega} = 0$. Solving the equation, we get $\omega = \pi/4$.