Multi-rate Signal Processing

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Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu. The LaTeX slides were made by Prof. Min Wu and Mr. Wei-Hong Chuang.

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Outline of Part-I: Multi-rate Signal Processing

Ref: Vaidyanathan tutorial paper (Proc. IEEE '90);
Book §1, §4, §5.

Single-rate v.s. Multi-rate Processing

- **Single-rate processing**: the digital samples before and after processing correspond to the same sampling frequency with respect to (w.r.t.) the analog counterpart.

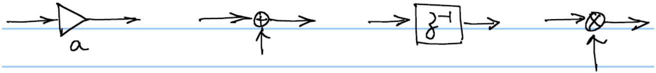
  e.g.: LTI filtering can be characterized by the freq. response.

- **The need of multi-rate**:
  - fractional sampling rate conversion in all-digital domain:
    e.g. 44.1kHz CD rate $\Longleftrightarrow$ 48kHz studio rate

- **The advantages of multi-rate signal processing**:
  - Reduce storage and computational cost
    - e.g.: polyphase implementation
  - Perform the processing in all-digital domain
    without using analog as an intermediate step that can:
    - bring inaccuracies – not perfectly reproducible
    - increase system design / implementation complexity

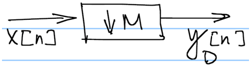
Basic Multi-rate Operations: Decimation and Interpolation

- Building blocks for traditional single-rate digital signal processing: multiplier (with a constant), adder, delay, multiplier (of 2 signals)

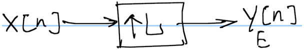


- New building blocks in multi-rate signal processing:

M-fold decimator



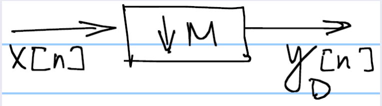
L-fold expander

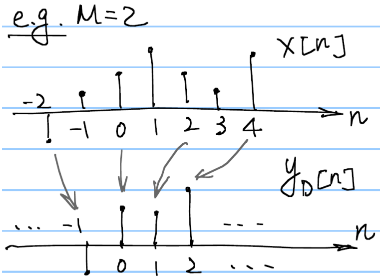


Readings: Vaidyanathan Book §4.1; tutorial Sec. II A, B

M-fold Decimator

$$y_D[n] = x[Mn], M \in \mathbb{N}$$





Corresponding to the physical time scale, it is as if we sampled the original signal in a slower rate when applying decimation.

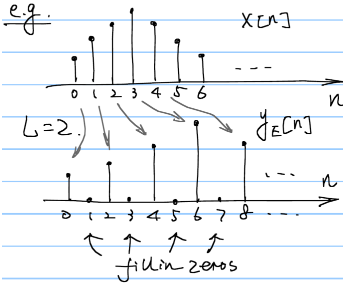
Questions:

- What potential problem will this bring?
- Under what conditions can we avoid it?
- Can we recover $x[n]$?

L-fold Expander

$$y_E[n] = \begin{cases} x[n/L] & \text{if } n \text{ is integer multiple of } L \in \mathbb{N} \\ 0 & \textit{otherwise} \end{cases}$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y_E[n]$$



Question: Can we recover $x[n]$ from $y_E[n]$? $\rightarrow$ Yes.

The expander does not cause loss of information.

Question: Are $\uparrow L$ and $\downarrow M$ <u>linear</u> and <u>shift invariant</u>?

Transform-Domain Analysis of Expanders

Derive the Z-Transform relation between the Input and Output:

(details)

Input-Output Relation on the Spectrum

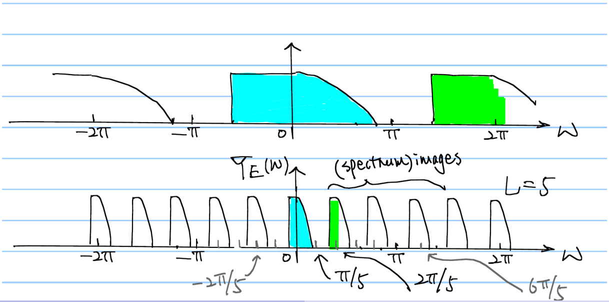
$$\mathbb{Y}_E(z) = \mathbb{X}(z^L)$$

(details)

Evaluating on the unit circle, the Fourier Transform relation is:

$$\mathbb{Y}_E(e^{j\omega}) = \mathbb{X}(e^{j\omega L}) \quad \Rightarrow \quad \mathbb{Y}_E(\omega) = \mathbb{X}(\omega L)$$

i.e. L-fold compressed version of $\mathbb{X}(\omega)$ along ω



Periodicity and Spectrum Image

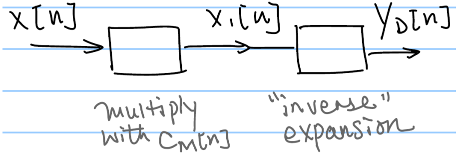
The Fourier Transform of a discrete-time signal has period of 2π.
With expander, $\mathbb{X}(\omega L)$ has a period of $2\pi/L$.

The multiple copies of the compressed spectrum over one period of 2π are called images.

And we say the expander creates an imaging effect.

Transform-Domain Analysis of Decimators

$$\mathbb{Y}_D(z) = \sum_{n=-\infty}^{\infty} y_D[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[nM]z^{-n}$$



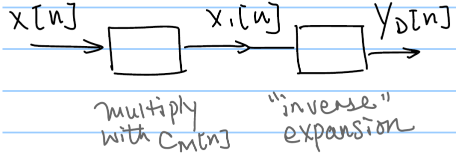
Define $x_1[n] = \begin{cases} x[n] & \text{if } n \text{ is integer multiple of } M \\ 0 & O.W. \end{cases}$, then we have

$$\mathbb{Y}_D(z) = \mathbb{X}_1(z^{\frac{1}{M}})$$ (details)

$$\mathbb{X}_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} \mathbb{X}(W_M^k z)$$ (details)

Transform-Domain Analysis of Decimators

$$\mathbb{Y}_D(z) = \sum_{n=-\infty}^{\infty} y_D[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[nM] z^{-n}$$



Putting all together:

$$\mathbb{Y}_D(z) = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{X}(W_M^k z^{\frac{1}{M}})$$

(details)

$$\mathbb{Y}_D(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{X}\left(\frac{\omega - 2\pi k}{M}\right)$$

(details)

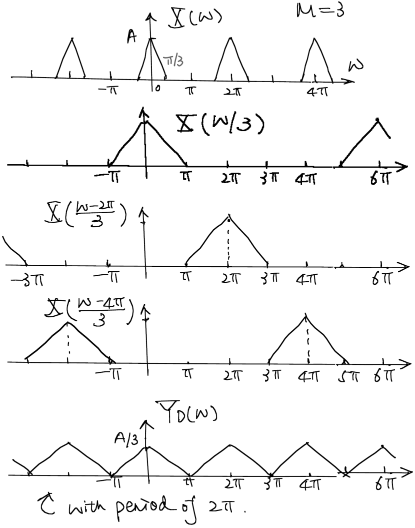
Frequency-Domain Illustration of Decimation

Interpretation of $\mathbb{Y}_D(\omega)$

Step-1: stretch $\mathbb{X}(\omega)$ by a factor of M to obtain $\mathbb{X}(\omega/M)$

Step-2: create $M-1$ copies and shift them in successive amounts of 2π

Step-3: add all M copies together and multiply by $1/M$.



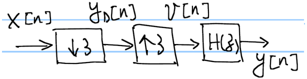
Aliasing

- The stretched version $\mathbb{X}(\omega/M)$ can in general overlap with its shifted replicas. This overlap effect is called aliasing.
- When aliasing occurs, we cannot recover $x[n]$ from the decimated version $y_D[n]$, i.e. $\downarrow M$ can be a lossy operation.
- We can avoid aliasing by limiting the bandwidth of $x[n]$ to

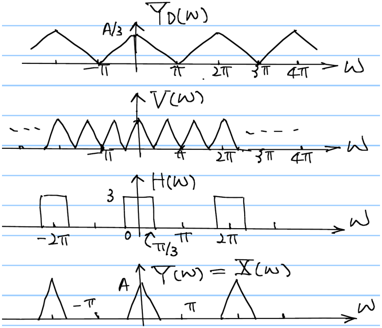
$$|\omega| < \pi/M.$$

- When no aliasing, we can recover $x[n]$ from the decimated version $y_D[n]$ by using an expander, followed by filtering of the unwanted spectrum images.

Example of Recovery from Decimated Signal



$y[n] = x[n]$ where no aliasing occurs.



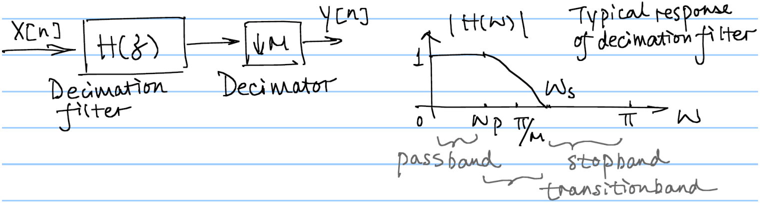
freq.-domain interpretation

Question: Is the bandlimit condition $|\omega| < \pi/M$ necessary?
What if $X(\omega)$ has a support over $[\pi/3, \pi]$ for $M = 3$?

Decimation Filters

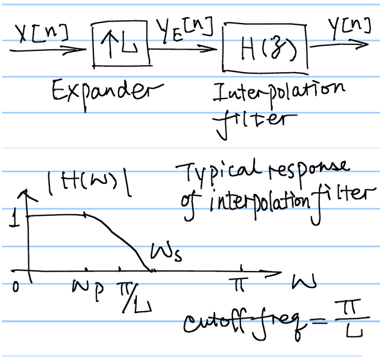
The decimator is normally preceded by a lowpass filter called decimator filter.

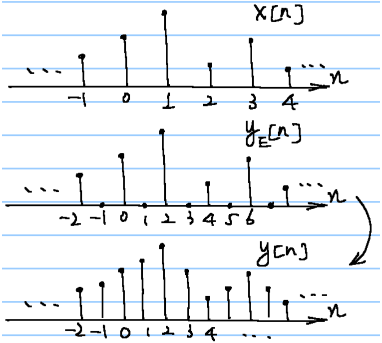
Decimator filter ensures the signal to be decimated is bandlimited and controls the extent of aliasing.



Interpolation Filters

An interpolation filter normally follows an expander to suppress all the images in the spectrum.





time-domain interpretation

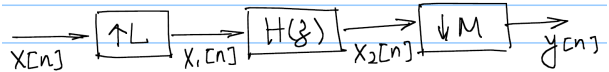
Fractional Sampling Rate Conversion

So far, we have learned how to increase or decrease sampling rate in the digital domain by integer factors.

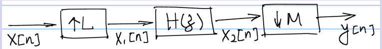
Question: How to change the rate by a rational fraction L/M?
(e.g.: audio 44.1kHz $\Longleftrightarrow$ 48kHz)

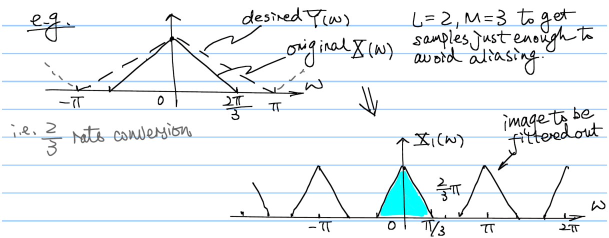
- Method-1: convert into an analog signal and resample
- Method-2: directly in digital domain by judicious combination of interpolation and decimation

Question: Decimate first or expand first? And why?

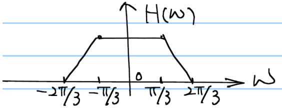


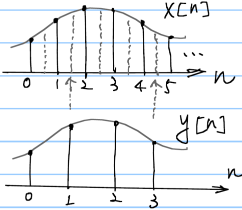
Fractional Rate Conversion





Use a low pass filter with passband greater than $\pi/3$ and stopband edge before $2\pi/3$ to remove images



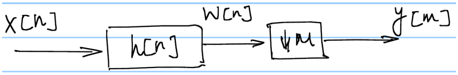


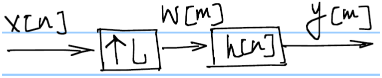
Equiv. to getting 2 samples out of every 3 original samples

- the signal now is critically sampled
- some samples kept are interpolated from $x[n]$

Time Domain Descriptions of Multirate Filters

Recall:

1. 

2. 

Summary of Time Domain Description

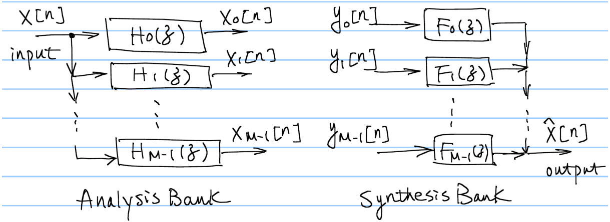
Input-output relation in the time domain for three types of multirate filters:

$$y[n] = \begin{cases} \sum_{k=-\infty}^{\infty} x[k]h[nM-k] & \text{M-fold decimation filter} \\ \sum_{k=-\infty}^{\infty} x[k]h[n-kL] & \text{L-fold interpolation filter} \\ \sum_{k=-\infty}^{\infty} x[k]h[nM-kL] & \text{M/L-fold decimation filter} \end{cases}$$

Note: Systems involving expander and decimator (plus filters) are in general linear time-varying (LTV) systems.

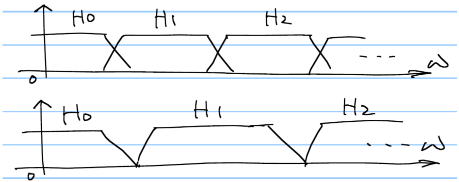
Digital Filter Banks

A digital filter bank is a collection of digital filters, with a common input or a common output.



- $H_i(z)$: analysis filters
- $x_k[n]$: subband signals
- $F_i(z)$: synthesis filters
- SIMO vs. MISO

- Typical frequency response for analysis filters:



Can be

- marginally overlapping
- non-overlapping
- (substantially) overlapping

Review: Discrete Fourier Transform

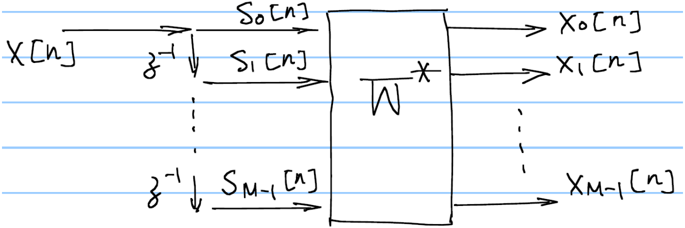
Recall:	time-domain	$\Rightarrow$	frequency-domain
M-point DFT	discrete periodic		discrete periodic

$$\begin{cases} \text{DFT: } \mathbb{X}[k] = \sum_{n=0}^{M-1} x[n] W^{nk} \\ \text{IDFT: } x[n] = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{X}[k] W^{-nk} \end{cases} \qquad (W = e^{-j2\pi/M})$$

- Subscript is often dropped from W_M if context is clear
- The $M \times M$ DFT matrix $\mathbf{W}$ is defined as $[\mathbf{W}]_{kn} = W^{kn}$
- We use $\mathbf{W}^*$ to represent the conjugate of $\mathbf{W}$; also note $\mathbf{W} = \mathbf{W}^T$ (symmetric)
- indexing convention in signal vector: $[x[0], x[1], ...]^T$, i.e. oldest first

DFT Filter Bank

Consider passing $x[n]$ through a delay chain
to get M sequences $\{s_i[n]\}$: $s_i[n] = x[n-i]$



i.e., treat $\{s_i[n]\}$ as a vector $\underline{s}[n]$, then apply $\mathbf{W}^*\underline{s}[n]$ to get $\underline{x}[n]$.
($\mathbf{W}^*$ instead of $\mathbf{W}$ due to newest component first in signal vector)

Question: What are the equiv. analysis filters?
And if having a multiplicative factor α_i to the $s_i[n]$?

Input-Output Relation of DFT Filter Bank

(details)

Relation between $H_i(z)$

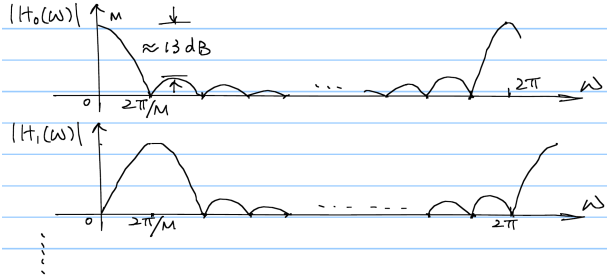
(details)

Uniform DFT Filter Bank

A filter bank in which the filters are related by

$$H_k(z) = H_0(zW^k)$$

is called a uniform DFT filter bank.



The response of filters $|H_k(\omega)|$ have a large amount of overlap.

Time-domain Interpretation of the Uniform DFT FB

(details)

Time-domain Interpretation of the Uniform DFT FB

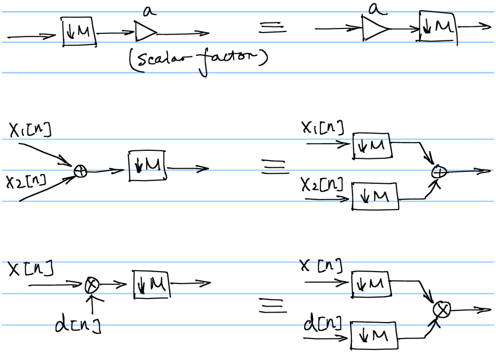
The DFT filter bank can be thought of as a spectrum analyzer

- The output $\{x_k[n]\}_{k=0}^{M-1}$ is the spectrum captured based on the most recent M samples of the input sequence $x[n]$.
- The filters themselves are not very good: wide transition bands and poor stopband attenuation of only 13dB
  – due to the simple rectangular sliding window $H_0(z)$.

Question: How can we improve the filters in the uniform DFT filter bank, esp. the prototype filter $H_0(z)$?

Interconnection of Building Blocks: Basic Properties

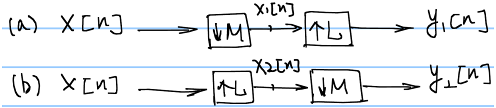
Basic interconnection properties:



} by the linearity of $\downarrow M$ & $\uparrow L$

Readings: Vaidyanathan Book §4.2; tutorial Sec. II B

Decimator-Expander Cascades



Questions:

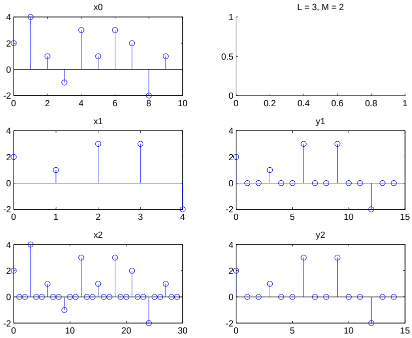
1. Is $y_1[n]$ always equal to $y_2[n]$? Not always.

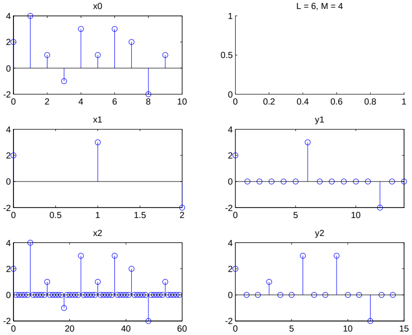
   E.g., when $L = M$, $y_2[n] = x[n]$, but

   $y_1[n] = x[n] \cdot c_M[n] \neq y_2[n]$, where $c_M[n]$ is a comb sequence

2. Under what conditions $y_1[n] = y_2[n]$?

Example of Decimator-Expander Cascades





Condition for $y_1[n] = y_2[n]$

Examine the ZT of $y_1[n]$ and $y_2[n]$: (details)

Condition for $y_1[n] = y_2[n]$

Equiv. to examine the condition of $\left\{W_M^k\right\}_{k=0}^{M-1} \equiv \left\{W_M^{kL}\right\}_{k=0}^{M-1}$:

iff M and L are relatively prime.

Question: Prove it. (see homework).

$\Rightarrow$ Thus the outputs of the two decimator-expander cascades, $\mathbb{Y}_1(z)$ and $\mathbb{Y}_2(z)$, are identical and $(a) \equiv (b)$ **iff** M **and** L **are relatively prime**.

The Noble Identities

Recall: the cascades of decimators and expanders with LTI systems appeared in decimation and interpolation filtering.

Question:

$$\to \boxed{H(z)} \to \boxed{\downarrow M} \to \quad ; \quad \to \boxed{\uparrow L} \to \boxed{H(z)} \to$$

$$\overset{?}{\equiv} \qquad\qquad\qquad\qquad \overset{?}{\equiv}$$

$$\to \boxed{\downarrow M} \to \boxed{H(z)} \to \quad ; \quad \to \boxed{H(z)} \to \boxed{\uparrow L} \to$$

$\Rightarrow$ Generally "No".

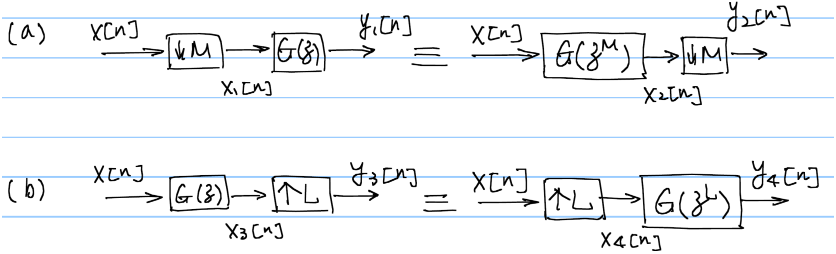
Observations:

① $\to z^{-1} \to \boxed{\downarrow M} \to \; \neq \; \to \boxed{\downarrow M} \to z^{-1} \to$ (for $M>1$) by shift variance.

② $\to z^{-M} \to \boxed{\downarrow M} \to \; = \; \to \boxed{\downarrow M} \to z^{-1} \to$; $\to \boxed{\uparrow L} \to z^{-L} \to \; = \; \to z^{-1} \to \boxed{\uparrow L} \to$

The Noble Identities

Consider a LTI digital filter with a transfer function $G(z)$:



Question: What kind of impulse response will a filter $G(z^L)$ have?

Recall: the transfer function $G(z)$ of a LTI digital filter is rational for practical implementation, i.e., a ratio of polynomials in z or z^{-1}. There should not be terms with fractional power in z or z^{-1}.

Proof of Noble Identities

details