Multi-rate Signal Processing
6. Quadrature Mirror Filter (QMF) Bank

Electrical & Computer Engineering
University of Maryland, College Park

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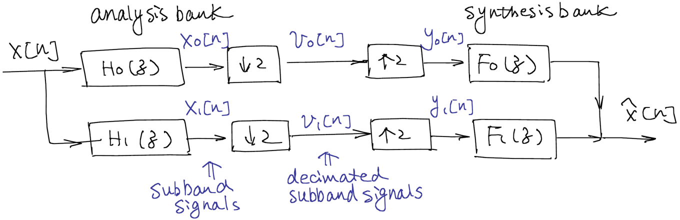
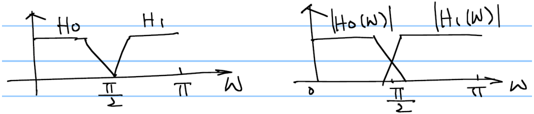
Contact: *minwu@umd.edu*. Updated: September 29, 2011.

Review: Two-channel Filter Bank

Recall: the 2-band QMF bank example in subband coding



Typical magnitude response



Overlapping filter response across $\pi/2$ may cause aliased subband signals

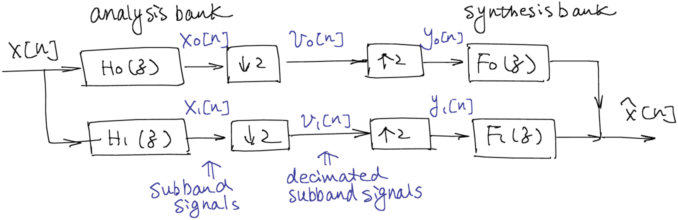
6.1 Errors Created in the QMF Bank

The reconstructed signal $\hat{x}[n]$ can differ from $x[n]$ due to

1. aliasing
2. amplitude distortion
3. phase distortion
4. processing of the decimated subband signal $v_k[n]$
   - quantization, coding, or other processing
   - inherent in practical implementation and/or depends on applications
     $\Rightarrow$ ignored in this section.

Readings: Vaidynathan Book 5.0-5.2; Tutorial Sec.VI.

Input-Output Relation



Examine the input-output relation: details

Input-Output Relation

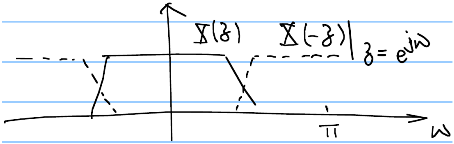
$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right]X(z) + \\ &\quad \frac{1}{2}\left[H_0(-z)F_0(z) + H_1(-z)F_1(z)\right]X(-z)\end{aligned}$$

In matrix-vector form: details

What is $X(-z)$?

- $X(-z)|_{z=e^{j\omega}} = X(\omega - \pi)$, i.e., shifted version of $X(\omega)$

  Referred to as the "alias term".



If $X(\omega)$ is not bandlimited by $\pi/2$, then $X(-z)$ may overlap with $X(z)$ spectrum.

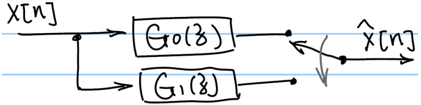
In the reconstructed signal $\hat{x}[n]$, this alias term reflects aliasing due to downsampling and residue imaging due to expansion.

Linear Periodically Time Varying (LPTV) Viewpoint

details Write $\hat{X}(z)$ expression as: $\hat{X}(z) = T(z)X(z) + A(z)X(-z)$

i.e., alternatingly taking output from one of the two LTI subsystems
(note: input and ouput have the same rate)

Linear Periodically Time Varying (LPTV) Viewpoint



If aliasing is cancelled (i.e., $A(z) = 0$), this will become LTI with transfer function $T(z)$.

Questions: Why we may want to permit some aliasing?

- To avoid excessive attenuation of input signal around $\omega = \frac{\pi}{2}$ and expensive $H_k(z)$ filters for sharp transition band, we permit some aliasing in the decimated analysis bank instead of trying to completely avoid it.
- We then choose synthesis filters so that the alias components in the two branches can cancel out each other.

Alias Cancellation

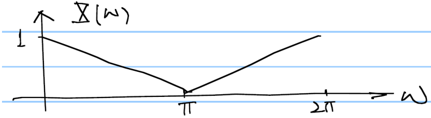
To cancel aliasing for all possible inputs $x[n]$ s.t.

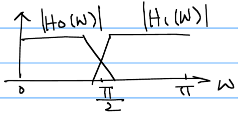
$$H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0,$$

we can choose

$$\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases} \qquad \text{(a sufficient condition)}$$

Example: sketch intermediate spectrums step-by-step

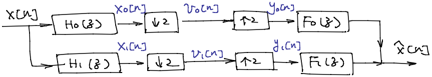


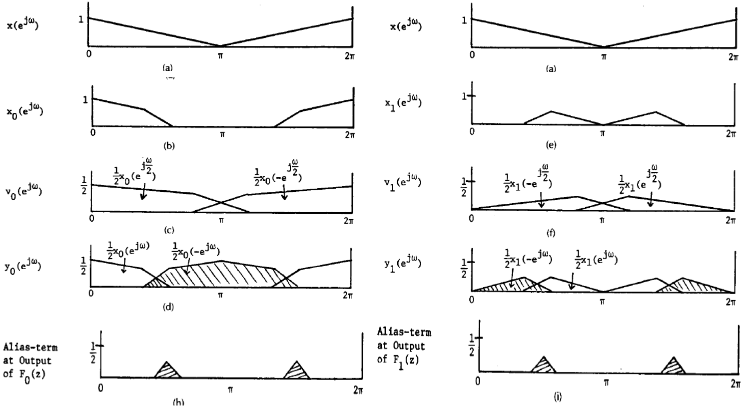


Alias Cancellation in the Spectrum

P.P. Vaidyanathan: "Multirate digital filters, filter banks, polyphase networks, andapplications: a tutorial", Proceedings of the IEEE, Jan 1990, Volume: 78, Issue: 1, pages 56-93. DOI: 10.1109/5.52200

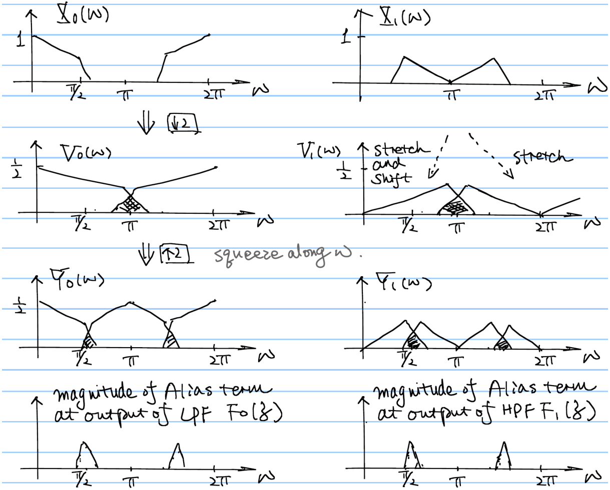
Fig. 23. Illustration of various Fourier transforms in two-channel QMF bank. Here horizontal axis represents ω. (a) Typical input. (b) Transform. (c) Aliasing effect. (d) Imaging effect. (e) Using x_1. (f) Using v_1. (g) Using y_1. (h) Alias-term at output of $F_0(z)$. (i) Alias-term at output of $F_1(z)$.





Alias Cancellation in the Spectrum (sketch)

Assume $H_0(z)$ and $H_1(z)$ have some overlap and across $\pi/2$



possible to choose $F_k(z)$ to make these terms cancel each other out

Amplitude and Phase Distortions

Distortion Transfer Function

For an aliasing-free QMF bank, $\hat{X}(z) = T(z)X(z)$,

where $T(z) = \frac{1}{2}\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right]$
$= \frac{1}{2}\left[H_0(z)H_1(-z) - H_1(z)H_0(-z)\right]$

This is called the distortion transfer function, or the overall transfer function of the alias-free system.

Let $T(\omega) = |T(\omega)|e^{j\phi(\omega)}$

To prevent amplitude distortion and phase distortion, $T(\omega)$ must be allpass (i.e. $|T(\omega)| = \alpha \neq 0$ for all ω, α is a constant) and linear phase (i.e., $\phi(\omega) = a + b\omega$ for constants a,b)

Properties of $T(z)$

- Perfect reconstruction (PR) property: if a QMF bank is free from aliasing, amplitude distortion and phase distortion, i.e., $T(z) = cz^{-n0} \Rightarrow \hat{x}[n] = cx[n - n_0]$

- With our above alias-free choice of $F_k(z)$, $T(z)$ is in the form of $T(z) = W(z) - W(-z)$, where $W(z) = H_0(z)H_1(-z)$.

  $\Rightarrow$ $T(z)$ has only odd power of z (as the even powers get cancelled), i.e., $T(z) = z^{-1}S(z^2)$ for some $S(z)$.

  So $|T(\omega)|$ has period of π (instead of 2π).
  And for real-coefficient filters, this implies $|T(\omega)|$ is symmetric w.r.t. $\pi/2$ for $0 \leq \omega < \pi$.

6.2 A Simple Alias-Free QMF System

Consider the analysis filters are related as

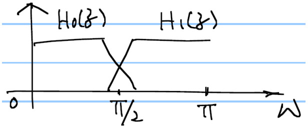
$$H_1(z) = H_0(-z)$$

For real filter coefficients, this means $|H_1(\omega)| = |H_0(\pi - \omega)|$.

$\because$ $|H_0(\omega)|$ symmetric w.r.t. $\omega = 0$; $|H_1(\omega)| \sim$ shift $|H_0(\omega)|$ by π.

i.e., $|H_1(\omega)|$ is a mirror image of $|H_0(\omega)|$ w.r.t. $\omega = \pi/2 = 2\pi/4$,
the "quadrature frequency" of the normalized sampling frequency.

If $H_0(z)$ is a good LPF,
then $H_1(z)$ is a good HPF.



(1) QMF Choice and Alias-free Condition

With QMF choice of $H_1(z) = H_0(-z)$, now the alias-free condition becomes

$$\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases} \Rightarrow \begin{cases} F_0(z) = H_0(z) \\ F_1(z) = -H_1(1z) \end{cases}$$

All four filters are completely determined by a single filter $H_0(z)$.

The distortion transfer function becomes

$$T(z) = \frac{1}{2}\left[H_0^2(z) - H_1^2(z)\right] = \frac{1}{2}\left[H_0^2(z) - H_0^2(-z)\right]$$

(2) Polyphase Representation of QMF

※ beneficial both computationally and conceptually

Let $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$ (Type-1 PD)

Then $H_1(z) = H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2)$

In matrix/vector form,

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix}$$

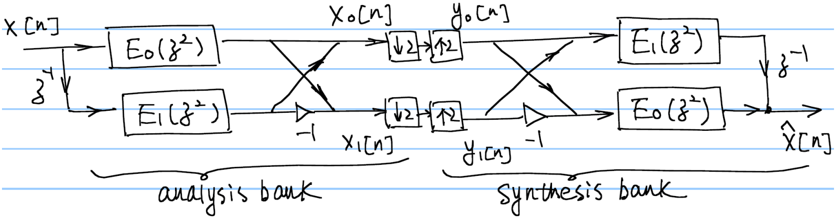
Similarly, for synthesis filters,

$$\begin{aligned} \begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} &= \begin{bmatrix} H_0(z) & -H_1(z) \end{bmatrix} \\ &= \begin{bmatrix} z^{-1}E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Polyphase Representation: Signal Flow Diagram

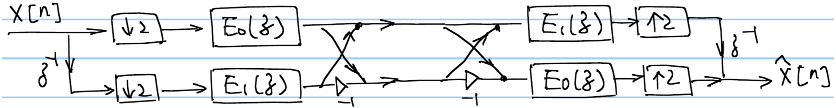
$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix}$$

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1}E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



Polyphase Representation: Efficient Structure

Rearrange using nobel identities to obtain efficient implementation:



For $H_0(z)$ of length $N \quad \Rightarrow \quad E_k(z)$ has length $N/2$

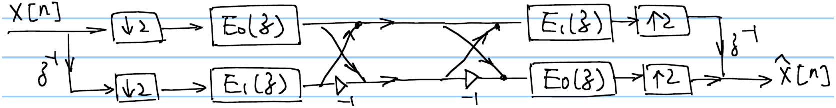
- Analysis bank: $N/2$ MPU, $N/2$ APU; same for synthesis bank
- Total: N MPU & APU

$\because H_0^2(z) = E_0^2(z^2) + E_1^2(z^2)z^{-2} + 2z^{-1}E_0(z^2)E_1^2(z^2)$

So the distortion transfer function becomes

$$T(z) = \frac{1}{2}\left[H_0^2(z) - H_0^2(-z)\right] = 2z^{-1}E_0(z^2)E_1(z^2)$$

Polyphase Representation: Matrix Form



In matrix form: (with MIMO transfer function for intermediate stages)

$$\underbrace{\begin{bmatrix} E_1(z) & 0 \\ 0 & E_0(z) \end{bmatrix}}_{\text{synthesis}} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \underbrace{\begin{bmatrix} E_0(z) & 0 \\ 0 & E_1(z) \end{bmatrix}}_{\text{analysis}}$$

$$= \begin{bmatrix} 2E_0(z)E_1(z) & 0 \\ 0 & 2E_0(z)E_1(z) \end{bmatrix}$$

※ Note: Multiplication is from left for each stage when intermediate signals are in column vector form.

Observations

The distortion transfer function of QMF

$T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$

- If $H_0(z)$ is FIR, so are $E_0(z), E_1(z)$ and $T(z)$.

- For $H_0(z)$ FIR and $H_1(z) = H_0(-z)$, the amplitude distortion can be eliminated **iff** $E_0(z)$ **and** $E_1(z)$ **represent a delay**:

$$\begin{cases} E_0(z) = c_0 z^{-n_0} \\ E_1(z) = c_1 z^{-n_1} \end{cases}$$

Observations

For $E_0(z)$ and $E_1(z)$ each representing a delay, we can only have analysis filters in the form of

$$\begin{cases} H_0(z) = c_0 z^{-2n_0} + c_1 z^{-(2n_1+1)} \\ H_1(z) = c_0 z^{-2n_0} - c_1 z^{-(2n_1+1)} \end{cases}$$

Such filters don't have sharp cutoff and good stopband attenuations.

Therefore $H_1(z) = H_0(-z)$ is not a good choice to build FIR perfect reconstruction QMF systems for such applications as subband coding.

(3) Eliminating Phase Distortions with FIR Filters

If $H_0(z)$ has linear phase, then we can show that

$$T(z) = \frac{1}{2}\left[H_0^2(z) - H_0^2(-z)\right]$$

also has linear phase (thus eliminating phase distortion).

Let $H_0(z) = \sum_{n=0}^{N} h_0[n]z^{-n}$ with $h_0[n]$ real. The linear phase and low pass conditions lead to $h_0[n] = h_0[N-n]$ (symmetric).

We can write $H_0(\omega) = e^{-j\omega\frac{N}{2}} \underbrace{R(\omega)}_{\text{real valued}}$

(3) Eliminating Phase Distortions with FIR Filters

$T(\omega)$ now becomes: details

Note: $|H_0(\omega)| = |R(\omega)|$ and $|H_0(\omega)|$ is even symmetric

$$\Rightarrow T(\omega) = \frac{e^{-j\omega N}}{2}[|H_0(\omega)|^2 - (-1)^N |H_0(\pi - \omega)|^2]$$

If N is even, $T(\omega)|_{\omega=\frac{\pi}{2}} = 0$, which brings severe amplitude distortion around $\omega = \pi/2$.

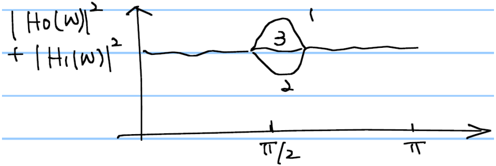
To avoid this, the filter order N should be **odd** (or length is even) so that $T(\omega) = \frac{e^{-j\omega N}}{2}\left[|H_0(\omega)|^2 + |H_0(\pi - \omega)|^2\right]$

(4) Minimizing Amplitude Distortion with FIR Filters

- Recall: after choosing $H_1(z) = H_0(-z)$, the amplitude distortion can be removed iff $H_0(z)$'s two polyphase components are pure delay.
  But such $H_0(z)$ doesn't have good low-pass response.
- For more flexible choices of $H_0(z)$ while eliminating aliasing and phase distortion, there will be some amplitude distortion.
- What we can do is to adjust the coefficients in $H_0(z)$ to minimize the amplitude distortion, i.e., to make $T(\omega)$ approximately constant:

$$|H_0(\omega)|^2 + |H_1(\omega)|^2 \approx 1$$

(4) Minimizing Amplitude Distortion with FIR Filters



1 – too much overlap b/w H_0 and H_1
2 – too little overlap
3 – good choice
(can be obtained by trial and error or by optimization formulation)

- Recall $T(z)$ has only odd power of z. For real-coeff. filter, $|T(\omega)|$ is symmetric w.r.t. $\pi/2$ for $0 \leq \omega < \pi$.
- By quadrature mirror condition, $|T(\omega)|$ is almost constant in the passbands of $H_0(z)$ and $H_1(z)$ if $H_0(z)$ has good passband and stopband responses.
- The main problem is with the transition band. The degree of overlap between $H_0(z)$ and $H_1(z)$ is crucial in determining this distortion.

See Vaidyanathan's Book §5.2.2 for details and examples

(5) Eliminating Amplitude Distortion with IIR Filters

How about IIR filters?

- The choice of $E_1(z) = \frac{1}{E_0(z)}$ can lead to perfect reconstruction and provide more room for designing $H(z)$.

  But the filters $H_k(z)$ would become IIR and may not provide desirable response.

- To completely eliminate amplitude distortion, $T(z)$ must be all-pass (which is IIR).

- Review: a 1st-order all-pass filter $G(z) = \frac{a^*+z^{-1}}{1+az^{-1}}$
  $\Rightarrow |G(\omega)| = 1$; zero $= -1/a^*$, pole $= -a$ (conjugate reciprocal).

(5) Eliminating Amplitude Distortion with IIR Filters

One way to make $T(z)$ allpass is to choose $E_0(z)$ and $E_1(z)$ to be IIR and allpass.

Let $E_0(z) = \frac{a_0(z)}{2}$ and $E_1(z) = \frac{a_1(z)}{2}$ where $a_0(z)$ and $a_1(z)$ are allpass with $|a_0(\omega)| = |a_0(\omega)| = 1$.

The analysis filter becomes
$H_0(z) = E_0(z^2) + z^{-1}E_1(z^2) = \frac{a_0(z^2)+z^{-1}a_1(z^2)}{2}$

$\Rightarrow$ possible to have good $H(\omega)$ response with such all-pass polyphase form.
Explore PPV book 5.3

The overall distortion transfer function is allpass:
$T(z) = \frac{z^{-1}}{2}a_0(z^2)a_1(z^2)$

Phase Distortion with IIR Filters

- This design of QMF bank is free from amplitude distortion and aliasing, regardless of the details of the allpass filters $a_0(z)$ and $a_1(z)$.
- But the phase distortion remains due to the IIR components. The phase distortion is governed by the phase responses of $a_0(z)$ and $a_1(z)$.

Question: Can $a_0(z)$ and $a_1(z)$ be designed to cancel out phase distortion?

Note the difficulty in designing filters to meet many constraints.

Summary

Many "wishes" to consider toward achieving alias-free P.R. QMF:
(0) alias free, (1) phase distortion, (2) amplitude distortion,
(3) desirable filter responses.

Can't satisfy them all at the same time, so often meet most of them and try to approximate/optimize the rest.

A particular relation of synthesis-analysis filters to cancel alias:

$$\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases} \quad \text{s.t. } H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0.$$

We considered a specific relation between the analysis filters:
$H1(z) = H_0(-z)$ s.t. response symmetric w.r.t. $\omega = \pi/2$ (QMF)

With polyphase structure: $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$

Summary: $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$

Case-1 $H_0(z)$ **is FIR:**

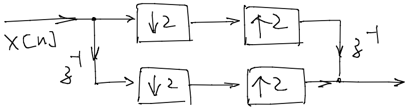
- P.R.: require polyphase components of $H_0(z)$ to be pure delay s.t. $H_0(z) = c_0 z^{-2n_0} + c_1 z^{-(2n_1+1)}$
  [cons] $H_0(\omega)$ response is very restricted.
- For more desirable filter response, the system may not be P.R., but can minimize distortion:
  - eliminate phase distortion: choose filter order N to be odd, and $h_0[n]$ be symmetric (linear phase)
  - minimize amplitude distortion: $|H_0(\omega)|^2 + |H_1(\omega)|^2 \approx 1$

Case-2 $H_0(z)$ **is IIR:**

- $E_1(z) = \frac{1}{E_0(z)}$ can get P.R. but restrict the filter responses.
- eliminate amplitude distortion: choose polyphase components to be all pass, s.t. $T(z)$ is all-pass, but may have some phase distortion

Look Ahead: Simple FIR P.R. Systems

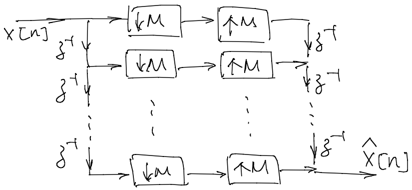
2-channel simple P.R. system:



How are $\hat{X}(z)$ and $X(z)$ related?

What are the equiv. $H_k(z)$ and $F_k(z)$?

Extend to M-channel:

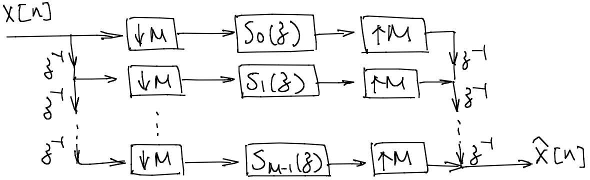


How are $\hat{X}(z)$ and $X(z)$ related?

What are the equiv. $H_k(z)$ and $F_k(z)$?

Interpretation: demultiplex then multiplex again

Look Ahead: Simple Filter Bank Systems

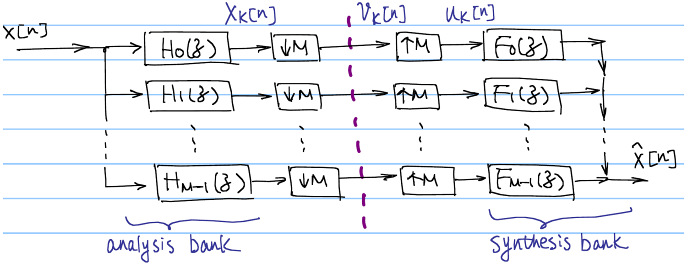


If all $S_k(z)$ are identical as $S(z)$, how are $\hat{X}(z)$ and $X(z)$ related?

How is this related to the simple M-channel P.R. system on the last page?

Look Ahead: *M*-channel filter bank

Study more general conditions of alias-free and PR; examine *M*-channel filter bank:



Derive the input-output relation. details

Input-Output Relation

Examine the input-output relation:

① Subband signals $X_k(z) = H_k(z)\, X(z)$ $\quad k=0,1.$

② Decimated subband signals $V_k(z) = [X_k(z)]\big|_{\downarrow 2}$

$$= \tfrac{1}{2} X_k(z^{1/2} W_2^0) + \tfrac{1}{2} X_k(z^{1/2} W_2^1)$$

recall $W_M \triangleq e^{-j\frac{2\pi}{M}}$.

$$= \tfrac{1}{2} X_k(z^{1/2}) + \tfrac{1}{2} X_k(-z^{1/2}) \quad k=0,1.$$

aliasing occurs if this part's spectrum overlaps with $X_k(z^{1/2})$

③ $Y_k(z) = V_k(z^2) = \tfrac{1}{2} X_k(z) + \tfrac{1}{2} X_k(-z)$

$$= \tfrac{1}{2} H_k(z) X(z) + \tfrac{1}{2} H_k(-z) X(-z) \quad k=0,1.$$

④ $\hat{X}(z) = F_0(z) Y_0(z) + F_1(z) Y_1(z)$

$$= \tfrac{1}{2}[H_0(z) F_0(z) + H_1(z) F_1(z)]\, X(z)$$
$$+ \tfrac{1}{2}[H_0(-z) F_0(z) + H_1(-z) F_1(z)]\, X(-z)$$

Input-Output Relation

In matrix-vector form:

$$2\hat{X}(z) = [X(z) \;\; X(-z)] \underbrace{\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}}_{\text{Define as } \mathcal{H}(z)} \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}$$

"alias component matrix"

$$= [F_0(z) \;\; F_1(z)] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

LPTV (Linear Periodically Time Varying) Viewpoint

Write $\hat{X}(z)$ expression as:

$$\hat{X}(z) = T(z)X(z) + A(z)X(-z)$$

$$\Leftrightarrow \quad \hat{x}[n] = \sum_{k} \left(t[k] + (-1)^{n-k} a[k]\right) x[n-k]$$

Define
$$\begin{cases} g_0[k] = t[k] + (-1)^k a[k] \\ g_1[k] = t[k] - (-1)^k a[k] \end{cases}$$

$$\Rightarrow \quad \hat{x}[n] = \begin{cases} g_0[n] * x[n] & n \text{ is even} \\ g_1[n] * x[n] & n \text{ is odd} \end{cases}$$

$$\begin{cases} X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \\ X(-z) = \sum_{n=-\infty}^{+\infty} x[n] \cdot (-1)^{-n} \cdot z^{-n} \end{cases}$$

i.e., alternatingly taking output from one of the two LTI subsystems (note: input and ouput have the same rate)

Eliminating Phase Distortions with FIR Filters

$T(\omega)$ now becomes

$$T(\omega) = \frac{1}{2}\left[H_0^2(\omega) - H_0^2(\omega-\pi) \right]$$
$$= \frac{1}{2}\left[e^{-j\omega N} R^2(\omega) - e^{-j(\omega-\pi)N} R^2(\omega-\pi) \right]$$
$$= \frac{e^{-j\omega N}}{2}\left[R^2(\omega) - (-1)^N R^2(\pi-\omega) \right]$$
$$= \frac{e^{-j\omega N}}{2}\left[|H_0(\omega)|^2 - (-1)^N |H_0(\pi-\omega)|^2 \right]$$

also used here $|H_0(\omega)| = |R(\omega)|$ and $|H_0(\omega)|$ being even symmetric

If N is even, $T(\omega)|_{\omega=\frac{\pi}{2}} = 0$, which brings severe amplitude distortion around $\omega = \pi/2$.

To avoid this, N should be odd so that $T(\omega) = \frac{e^{-j\omega N}}{2}\left[|H_0(\omega)|^2 + |H_0(\pi-\omega)|^2 \right]$