Multi-rate Signal Processing

Prof. Min Wu

University of Maryland, College Park

minwu@umd.edu

Updated: September 19, 2011

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu.

The LaTeX version includes contributions from Mr. Wei-Hong Chuang.

Outline of Part-I: Multi-rate Signal Processing

§1.1 Building blocks and their properties
§1.2 Properties of interconnection of multi-rate building blocks
§1.3 Polyphase representation
§1.4 Multistage implementation
§1.5 Applications (brief): digital audio system; subband coding
§1.6 Quadrature mirror filter bank (2-channel)
§1.7 M-channel filter bank
§1.8 Perfect reconstruction filter bank
§1.9 Aliasing free filter banks
§1.10 Application: multiresolution analysis

Ref: Vaidyanathan tutorial paper (Proc. IEEE '90);
Book §1, §4, §5.

Single-rate v.s. Multi-rate Processing

- **Single-rate processing**: the digital samples before and after processing correspond to the same sampling frequency with respect to (w.r.t.) the analog counterpart.

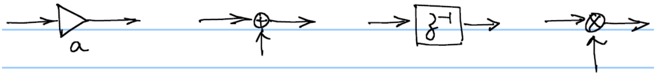
  e.g.: LTI filtering can be characterized by the freq. response.

- **The need of multi-rate**:
  - fractional sampling rate conversion in all-digital domain:
    e.g. $44.1k$HZ CD rate $\Longleftrightarrow$ $48k$HZ studio rate

- **The advantages of multi-rate signal processing**:
  - Reduce storage and computational cost
    - e.g.: polyphase implementation
  - Perform the processing in all-digital domain
    without using analog as an intermediate step that can:
    - bring inaccuracies – not perfectly reproducible
    - increase system design / implementation complexity

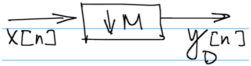
Basic Multi-rate Operations: Decimation and Interpolation

- Building blocks for traditional single-rate digital signal processing: multiplier (with a constant), adder, delay, multiplier (of 2 signals)

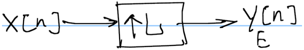


- New building blocks in multi-rate signal processing:

M-fold decimator



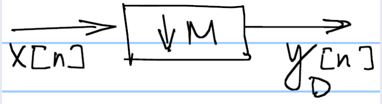
L-fold expander

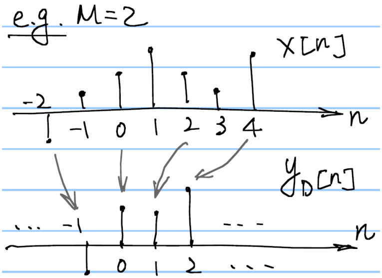


Readings: Vaidyanathan Book §4.1; tutorial Sec. II A, B

M-fold Decimator

$y_D[n] = x[Mn], M \in \mathbb{N}$





Corresponding to the physical time scale, it is as if we sampled the original signal in a slower rate when applying decimation.

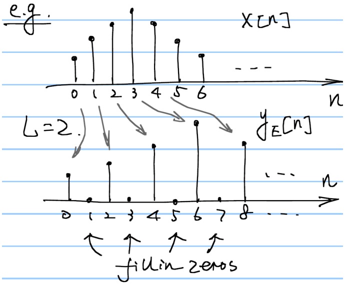
Questions:

- What potential problem will this bring?
- Under what conditions can we avoid it?
- Can we recover $x[n]$?

L-fold Expander

$$y_E[n] = \begin{cases} x[n/L] & \text{if } n \text{ is integer multiple of } L \in \mathbb{N} \\ 0 & \textit{otherwise} \end{cases}$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow y_E[n]$$



Question: Can we recover $x[n]$ from $y_E[n]$? $\rightarrow$ Yes.

The expander does not cause loss of information.

Question: Are $\uparrow L$ and $\downarrow M$ <u>linear</u> and <u>shift invariant</u>?

Transform-Domain Analysis of Expanders

Derive the Z-Transform relation between the Input and Output:

(details)

Input-Output Relation on the Spectrum

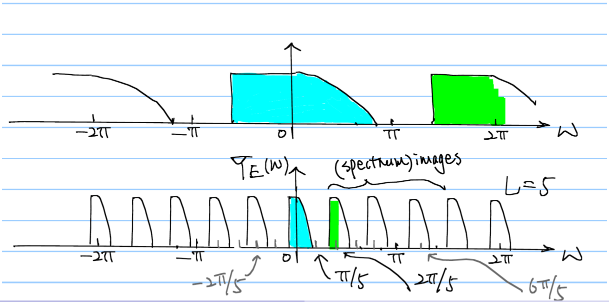
$$\mathbb{Y}_E(z) = \mathbb{X}(z^L)$$

(details)

Evaluating on the unit circle, the Fourier Transform relation is:

$$\mathbb{Y}_E(e^{j\omega}) = \mathbb{X}(e^{j\omega L}) \quad \Rightarrow \quad \mathbb{Y}_E(\omega) = \mathbb{X}(\omega L)$$

i.e. L-fold compressed version of $\mathbb{X}(\omega)$ along ω



Periodicity and Spectrum Image

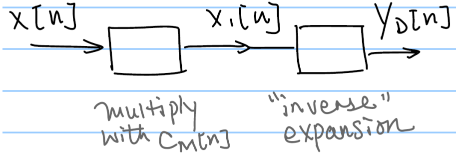
The Fourier Transform of a discrete-time signal has period of 2π.
With expander, $\mathbb{X}(\omega L)$ has a period of $2\pi/L$.

The multiple copies of the compressed spectrum over one period of 2π are called images.

And we say the expander creates an imaging effect.

Transform-Domain Analysis of Decimators

$$\mathbb{Y}_D(z) = \sum_{n=-\infty}^{\infty} y_D[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[nM] z^{-n}$$



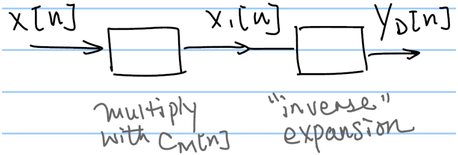
Define $x_1[n] = \begin{cases} x[n] & \text{if } n \text{ is integer multiple of } M \\ 0 & O.W. \end{cases}$, then we have

$\mathbb{Y}_D(z) = \mathbb{X}_1(z^{\frac{1}{M}})$ (details)

$\mathbb{X}_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{X}(W_M^k z)$ (details)

Transform-Domain Analysis of Decimators

$$\mathbb{Y}_D(z) = \sum_{n=-\infty}^{\infty} y_D[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[nM] z^{-n}$$



Putting all together:

$$\mathbb{Y}_D(z) = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{X}(W_M^k z^{\frac{1}{M}})$$

(details)

$$\mathbb{Y}_D(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{X}\left(\frac{\omega - 2\pi k}{M}\right)$$

(details)

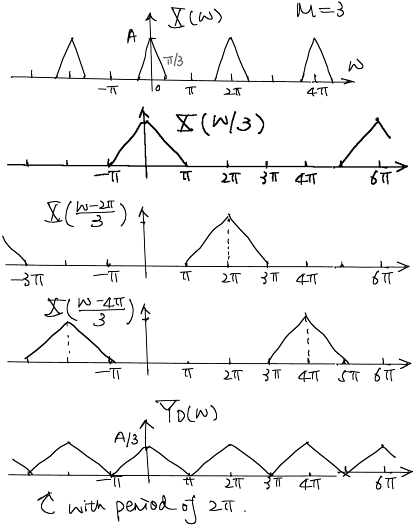
Frequency-Domain Illustration of Decimation

Interpretation of $\mathbb{Y}_D(\omega)$

Step-1: stretch $\mathbb{X}(\omega)$ by a factor of M to obtain $\mathbb{X}(\omega/M)$

Step-2: create $M-1$ copies and shift them in successive amounts of 2π

Step-3: add all M copies together and multiply by $1/M$.



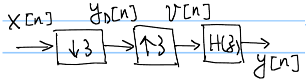
Aliasing

- The stretched version $\mathbb{X}(\omega/M)$ can in general overlap with its shifted replicas. This overlap effect is called aliasing.
- When aliasing occurs, we cannot recover $x[n]$ from the decimated version $y_D[n]$, i.e. $\downarrow M$ can be a lossy operation.
- We can avoid aliasing by limiting the bandwidth of $x[n]$ to

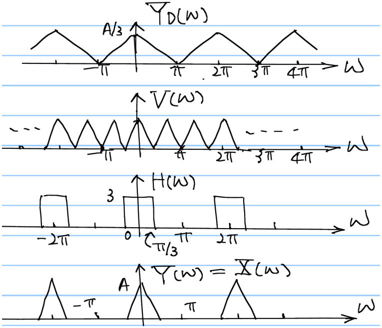
$$|\omega| < \pi/M.$$

- When no aliasing, we can recover $x[n]$ from the decimated version $y_D[n]$ by using an expander, followed by filtering of the unwanted spectrum images.

Example of Recovery from Decimated Signal



$y[n] = x[n]$ where no aliasing occurs.



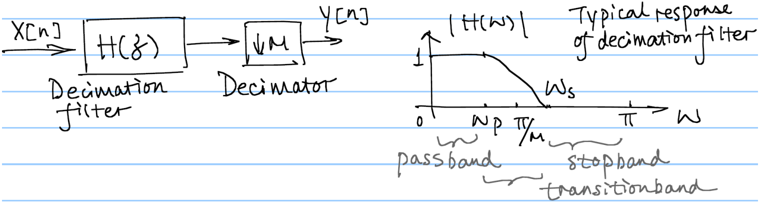
freq.-domain interpretation

Question: Is the bandlimit condition $|\omega| < \pi/M$ necessary?

Decimation Filters

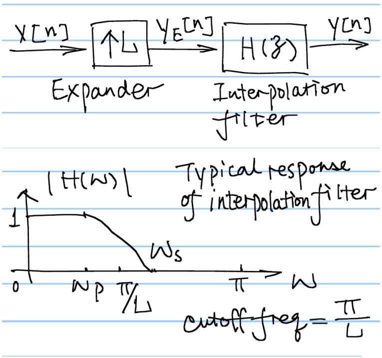
The decimator is normally preceded by a lowpass filter called decimator filter.

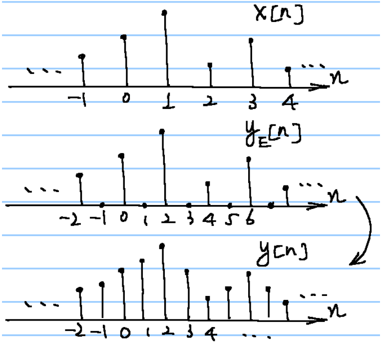
Decimator filter ensures the signal to be decimated is bandlimited and controls the extent of aliasing.



Interpolation Filters

An interpolation filter normally follows an expander to suppress all the images in the spectrum.





time-domain interpretation

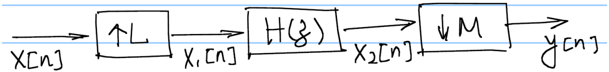
Fractional Sampling Rate Conversion

So far, we have learned how to increase or decrease sampling rate in the digital domain by integer factors.

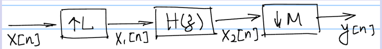
Question: How to change the rate by a rational fraction L/M?
(e.g.: audio 44.1k $\Longleftrightarrow$ 48k)

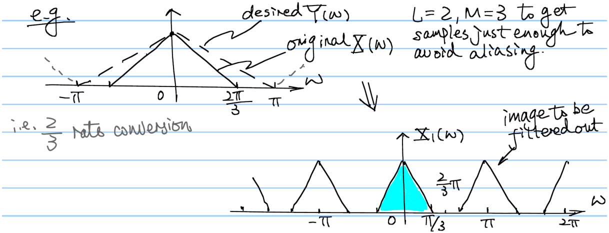
- Method-1: convert into an analog signal and resample
- Method-2: directly in digital domain by judicious combination of interpolation and decimation

Question: Decimate first or expand first? And why?

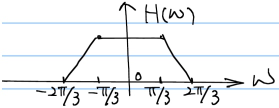


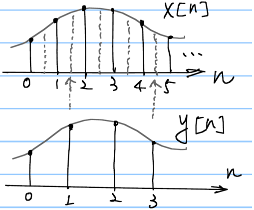
Fractional Rate Conversion





Use a low pass filter with passband greater than $\pi/3$ and stopband edge before $2\pi/3$ to remove images



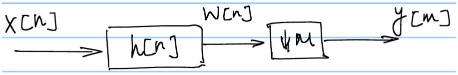


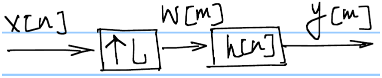
Equiv. to getting 2 samples out of every 3 original samples

- the signal now is critically sampled
- some samples kept are interpolated from $x[n]$

Time Domain Descriptions of Multirate Filters

Recall:

1. 

2. 

Summary of Time Domain Description

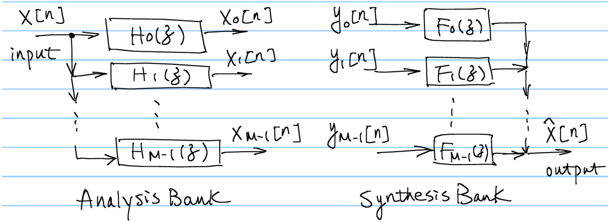
Input-output relation in the time domain for three types of multirate filters:

$$y[n] = \begin{cases} \sum_{k=-\infty}^{\infty} x[k]h[nM-k] & \text{M-fold decimation filter} \\ \sum_{k=-\infty}^{\infty} x[k]h[n-kL] & \text{L-fold interpolation filter} \\ \sum_{k=-\infty}^{\infty} x[k]h[nM-kL] & \text{M/L-fold decimation filter} \end{cases}$$

Note: Systems involving expander and decimator (plus filters) are in general linear time-varying (LTV) systems.

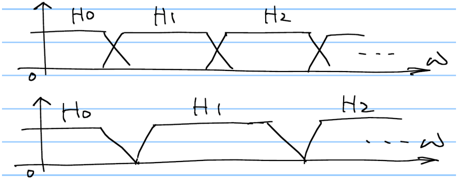
Digital Filter Banks

A digital filter bank is a collection of digital filters, with a common input or a common output.



- $H_i(z)$: analysis filters
- $x_k[n]$: subband signals
- $F_i(z)$: synthesis filters

- Typical frequency response for analysis filters: can be



- marginally overlapping
- non-overlapping
- (substantially) overlapping

Review: Discrete Fourier Transform

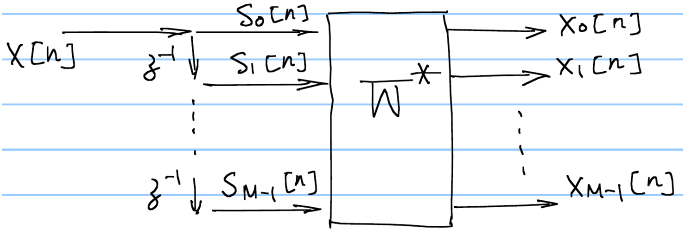
Recall:	time-domain	$\Rightarrow$	frequency-domain
M-point DFT	discrete periodic		discrete periodic

$$\begin{cases} \text{DFT: } \mathbb{X}[k] = \sum_{n=0}^{M-1} x[n] W^{nk} \\ \text{IDFT: } x[n] = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{X}[k] W^{-nk} \end{cases} \qquad (W = e^{-j2\pi/M})$$

- Subscript is often dropped from W_M if context is clear
- The $M \times M$ DFT matrix $\mathbf{W}$ is defined as $[\mathbf{W}]_{kn} = W^{kn}$
- We use $\mathbf{W}^*$ to represent the conjugate of $\mathbf{W}$; also note $\mathbf{W} = \mathbf{W}^T$ (symmetric)

DFT Filter Bank

Consider passing $x[n]$ through a delay chain
to get M sequences $\{s_i[n]\}$: $s_i[n] = x[n-i]$



i.e., treat $\{s_i[n]\}$ as a vector $\underline{s}[n]$, then apply $\mathbf{W}^* \underline{s}[n]$ to get $\underline{x}[n]$.

Question: What are the equiv. analysis filters?

Input-Output Relation of DFT Filter Bank

(details)

Relation between $H_i(z)$

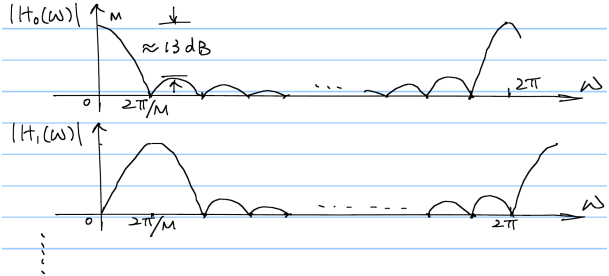
(details)

Uniform DFT Filter Bank

A filter bank in which the filters are related by

$$H_k(z) = H_0(zW^k)$$

is called a uniform DFT filter bank.



The response of filters $|H_k(\omega)|$ have a large amount of overlap.

Time-domain Interpretation of the Uniform DFT FB

(details)

Time-domain Interpretation of the Uniform DFT FB

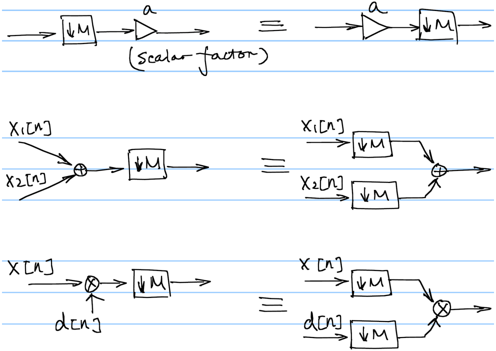
The DFT filter bank can be thought of as a spectrum analyzer

- The output $\{x_k[n]\}_{k=0}^{M-1}$ is the spectrum captured based on the most recent M samples of the input sequence $x[n]$.
- The filters themselves are not very good: wide transition bands and poor stopband attenuation of only 13dB
  – due to the simple rectangular sliding window $H_0(z)$.

Question: How can we improve the filters in the uniform DFT filter bank?

Interconnection of Building Blocks: Basic Properties

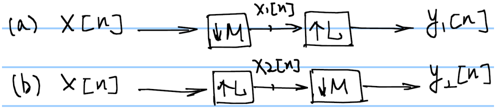
Basic interconnection properties:



} by the linearity of $\downarrow M$ & $\uparrow L$

Readings: Vaidyanathan Book §4.2; tutorial Sec. II B

Decimator-Expander Cascades



Questions:

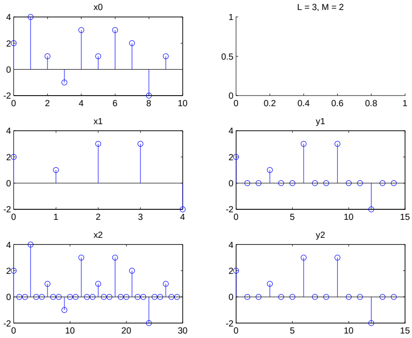
1. Is $y_1[n]$ always equal to $y_2[n]$? Not always.

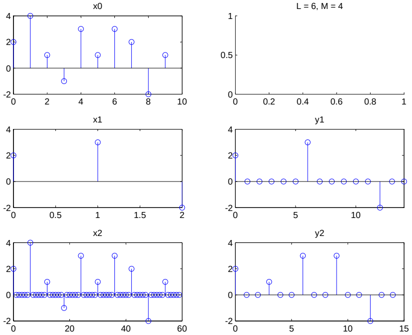
   E.g., when $L = M$, $y_2[n] = x[n]$, but

   $y_1[n] = x[n] \cdot c_M[n] \neq y_2[n]$, where $c_M[n]$ is a comb sequence

2. Under what conditions $y_1[n] = y_2[n]$?

Example of Decimator-Expander Cascades





Condition for $y_1[n] = y_2[n]$

Examine the ZT of $y_1[n]$ and $y_2[n]$: (details)

Condition for $y_1[n] = y_2[n]$

Equiv. to examine the condition of $\left\{W_M^k\right\}_{k=0}^{M-1} \equiv \left\{W_M^{kL}\right\}_{k=0}^{M-1}$:

iff M and L are relatively prime.

Question: Prove it. (see homework).

$\Rightarrow$ Thus the outputs of the two decimator-expander cascades, $\mathbb{Y}_1(z)$ and $\mathbb{Y}_2(z)$, are identical and $(a) \equiv (b)$ **iff** M **and** L **are relatively prime**.

The Noble Identities

Recall: the cascades of decimators and expanders with LTI systems appeared in decimation and interpolation filtering.

Question:

$$\to \boxed{H(z)} \to \boxed{\downarrow M} \to \quad ; \quad \to \boxed{\uparrow L} \to \boxed{H(z)} \to$$

$$? \equiv \qquad\qquad\qquad ? \equiv$$

$$\to \boxed{\downarrow M} \to \boxed{H(z)} \to \quad ; \quad \to \boxed{H(z)} \to \boxed{\uparrow L} \to$$

$\Rightarrow$ Generally "No".

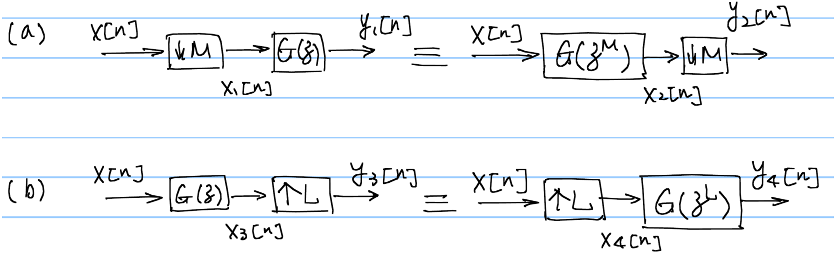
Observations:

① $\to z^{-1} \to \boxed{\downarrow M} \to \;\neq\; \to \boxed{\downarrow M} \to z^{-1} \to$ (for $M>1$) by shift variance.

② $\to z^{-M} \to \boxed{\downarrow M} \to \;=\; \to \boxed{\downarrow M} \to z^{-1} \to$; $\to \boxed{\uparrow L} \to z^{-L} \to \;=\; \to z^{-1} \to \boxed{\uparrow L} \to$

The Noble Identities

Consider a LTI digital filter with a transfer function $G(z)$:



Recall: the transfer function $G(z)$ of a LTI digital filter is rational for practical implementation, i.e., a ratio of polynomials in z or z^{-1}. There should not be terms with fractional power in z or z^{-1}.

Proof of Noble Identities

details

Detailed Derivations

Transform-Domain Analysis of Expanders

Z-Transform Relation between the Input and Output:

$$\mathbb{Y}_E(z) = \mathbb{X}(z^L)$$

Proof:

$$Y_E(z) = \sum_{n=-\infty}^{+\infty} y_E[n] z^{-n} = \sum_{n=kL,\, k\in\mathbb{Z}} y_E[n] z^{-n}$$

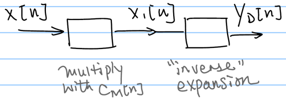
(i.e. exclude expanded zeros)

$$= \sum_{k=-\infty}^{+\infty} y_E[kL] z^{-kL}$$

$$= \sum_{k=-\infty}^{+\infty} x[k] (z^L)^{-k} = X(z^L)$$

Transform-Domain Analysis of Decimators

$$Y_D(z) = \sum_{n=-\infty}^{+\infty} y_D[n]\, z^{-n} = \sum_{n=-\infty}^{+\infty} x[nM]\, z^{-n}$$



Define $x_1[n] = \begin{cases} x[n] & \text{if } n \text{ is integer multiple of } M \\ 0 & \text{o.w.} \end{cases}$

We have $Y_D(z) = \sum_{k=nM,\, n\in\mathbb{Z}} x[k]\, z^{-k/M} = \sum_{k=-\infty}^{+\infty} x_1[k] \left(z^{1/M}\right)^{-k} = X_1(z^{1/M})$

change variable

Transform-Domain Analysis of Decimators

To establish the transform-domain relation between $X_1(z)$ and $X(z)$:

We note $x_1[n]$ can be written as

$$x_1[n] = c_M[n]\, x[n]$$

where $c_M[n] = \begin{cases} 1 & \text{if } n \text{ is integer multiple of } M \\ 0 & \text{o.w.} \end{cases}$

("comb" sequence)

$\boxed{\text{Trick}}$ Using the M^{th} root of unity W_M defined as

$$W_M = e^{-j2\pi/M}$$

We have

$$c_M[n] = \frac{1}{M}\sum_{k=0}^{M-1} W_M^{-kn}.$$

Transform-Domain Analysis of Decimators

By the definition of ZT:

$$X_1(z) = \sum_{n=-\infty}^{+\infty} x_1[n] z^{-n} = \sum_{n=-\infty}^{+\infty} c_M[n] x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{M} \sum_{k=0}^{M-1} W_M^{-kn} x[n] z^{-n}$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} \sum_{n=-\infty}^{+\infty} x[n] (z \cdot W_M^k)^{-n}$$

$$= \underbrace{\frac{1}{M} \sum_{k=0}^{M-1} X(W_M^k \cdot z)}$$

$$\therefore Y_D(z) = X_1(z^{1/M}) = \frac{1}{M} \sum_{k=0}^{M-1} X(W_M^k z^{1/M})$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(e^{-j2\pi k/M} z^{1/M})$$

Transform-Domain Analysis of Decimators

Fourier Spectrum: set $z = e^{j\omega}$

$$Y_D(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{-j2\pi k/M} e^{j\omega/M})$$

$$= \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M})$$

$$\therefore \; Y_D(\omega) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\omega - 2\pi k}{M}\right)$$

Time Domain Descriptions of Multirate Filters

Recall:

1.

$$x[n] \longrightarrow \boxed{h[n]} \xrightarrow{w[n]} \boxed{\downarrow M} \longrightarrow y[m]$$

$$w[n] = \sum_k h[k]\, x[n-k] = \sum_k h[n-k]\, x[k]$$

$$y[m] = w[Mm] = \sum_k h[k]\, x[Mm-k]$$

$$= \sum_k \underline{h[Mm-k]}\, x[k]$$

2.

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{w[m]} \boxed{h[n]} \longrightarrow y[m]$$

$$w[m] = \begin{cases} x[m/L] & \text{if } m \text{ is multiple of } L \\ 0 & \text{o.w.} \end{cases}$$

$$y[m] = \sum_k w[k]\, h[m-k] = \sum_{k'} x[k']\, h[m-k'L]$$

only keep non-zero $w[m]$
for all $k = k'L$

Input-Output Relation of DFT Filter Bank

k^{th} row of $\overline{W}^*$

$$X_k[n] = \sum_{i=0}^{M-1} S_i[n]\, W^{-ki} \sim M\text{-point IDFT of } \{S_0[n], \ldots S_{M-1}[n]\}$$

(except no factor $1/M$)

$$X_k(z) = \sum_{i=0}^{M-1} S_i(z)\, W^{-ki} \overset{\text{delay}}{=} \sum_{i=0}^{M-1} z^{-i} W^{-ki} X(z)$$

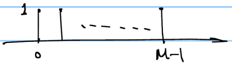
$$= \underbrace{\sum_{i=0}^{M-1} (W^k z)^{-i}}\, X(z)$$

Define this as $H_k(z)$, we have:

(the transfer functions)

Relation between $H_i(z)$

$$H_0(z) = \sum_{i=0}^{M-1} z^{-i}$$ i.e. ZT of a rectangular window



$$\xrightarrow{ZT} \quad H_0(z) = \frac{1 - z^{-M}}{1 - z^{-1}}$$

$$\longrightarrow \quad |H_0(\omega)| = \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right|$$

$$H_k(z) = H_0(W^k z) = H_0(e^{-j2\pi k/M} z)$$

$$\xrightarrow{z = e^{j\omega}} \quad H_k(\omega) = H_0\left(\omega - \frac{2\pi k}{M}\right)$$

i.e. uniformly shifted version of $H_0(\omega)$ spectrum

Time-domain Interpretation of the Uniform DFT FB

$$X_k[n+M-1] = \sum_{i=0}^{M-1} x[n+M-1-i]\, W^{-ki}$$

let $l = M-1-i$
$\Rightarrow i = M-1-l$

Here we consider a delayed version of x_k[ ] for convenience.

$$= \sum_{l=0}^{M-1} x[n+l]\, W^{kl} \cdot W^{-k(M-1)}$$

Note $W^M = 1$

$$= \underbrace{W^{k}}_{\text{phase shift}} \underbrace{\sum_{l=0}^{M-1} x[n+l]\, W^{kl}}_{k^{th}\text{ point of the } M\text{-point DFT of } \{x[n], \ldots, x[n+M-1]\}}$$

(linear phase term reflects time delay: $e^{-j\omega}\big|_{\omega=\frac{2\pi}{M}k}$)

Condition for $y_1[n] = y_2[n]$

Examine the ZT of $y_1[n]$ and $y_2[n]$:

$$\left.\begin{aligned} Y_1(z) &= X_1(z^L) \\ X_1(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^k z^{1/M}) \end{aligned}\right\} \Rightarrow Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^k z^{L/M})$$

$$\left.\begin{aligned} Y_2(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X_2(W_M^k z^{1/M}) \\ X_2(z) &= X(z^L) \end{aligned}\right\} \Rightarrow Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^{kL} z^{L/M})$$

need to raise the whole term to Lth power

$W_M^k = e^{-j\frac{2\pi k}{M}}$, $k=0, \ldots M-1$ are M distinct M^{th} roots of unity

$W_M^{kL} = e^{-j\frac{2\pi kL}{M}}$, $k=0, \ldots M-1$ may not represent M distinct numbers when L and M share common factors.

Proof of Noble Identities

Proof:

(a) $Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X_2(W_M^k z^{1/M})$

$X_2(z) = G(z^M) X(z) \Rightarrow X_2(W_M^k z^{1/M}) = G(W_M^{Mk} z) X(W_M^k z^{1/M})$

$\therefore Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} G(z) X(W_M^k z^{1/M})$

$= G(z) X_1(z) = Y_1(z)$

(b) $Y_4(z) = G(z^L) X_4(z) = G(z^L) X(z^L)$

$$\left.\begin{array}{l} Y_3(z) = X_3(z^L) \\ X_3(z) = G(z) X(z) \end{array}\right\} \Rightarrow \begin{array}{l} Y_3(z) = G(z^L) X(z^L) \\ \qquad = Y_4(z) \end{array}$$

∎

Equiv. to examine for $\left\{W_M^k\right\}_{k=0}^{M-1} \equiv ? \equiv \left\{W_M^{kL}\right\}_{k=0}^{M-1}$

$\left\{W_M^k\right\}_{k=0}^{M-1} \equiv \left\{W_M^{kL}\right\}_{k=0}^{M-1}$ iff M and L are <u>relatively prime</u>.

① "$\Rightarrow$". Prove by contradiction, i.e. let L and M have a common factor $\alpha \geqslant 2$ s.t. $M \equiv \alpha m$ for some integer $m < M$, and $L \equiv \alpha l$ for some $l < L$.

$mL = m\alpha l = M \cdot l \Rightarrow mL \bmod M = 0$

i.e. the set $\{0, L, 2L, \cdots (M-1)L\} \bmod M$ has at most $M-1$ distinct elements, thus $\neq \{0, 1, \cdots, (M-1)\}$.

<u>Contradict</u> with the given condition.

② "$\Leftarrow$". Prove by contradiction, i.e. suppose the two sets are different, where $\exists$ K_1 & K_2 s.t. $0 \leq K_2 < K_1 \leq M-1$ and $K_1 L \bmod M = K_2 L \bmod M$.

This means $\exists$ some integers a_1 and a_2, we have

$$K_1 L - a_1 M = K_2 L - a_2 M \text{ and } \begin{cases} (K_1-1)L < a_1 M \leq K_1 L \\ (K_2-1)L < a_2 M \leq K_2 L \end{cases}$$

$$\Downarrow$$

$$(K_1-K_2)L = (a_1-a_2)M \Rightarrow a_1 - a_2 = (K_1-K_2)\frac{L}{M}$$

$\because$ (a_1-a_2) is an integer $\therefore$ $(K_1-K_2)L$ should be multiples of M

$\because$ $0 < (K_1-K_2) \leq M-1$ $\therefore$ L must have some common factors with M.

$\Rightarrow$ <u>Contradiction</u>.

$\blacksquare$

Multi-rate Signal Processing
3. The Polyphase Representation

Prof. Min Wu

University of Maryland, College Park

minwu@umd.edu

Updated: September 19, 2011

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu.

The LaTeX version includes contributions from Mr. Wei-Hong Chuang.

Polyphase Representation: Basic Idea

Example: FIR filter $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

Group even and odd indexed coefficients, respectively:
$\Rightarrow H(z) = (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2})$,

More generally: Given a filter $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$, by grouping the odd and even numbered coefficients, we can write

$$H(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h[2n+1]z^{-2n}$$

Polyphase Representation: Definition

$$H(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h[2n+1]z^{-2n}$$

Define $E_0(z)$ and $E_1(z)$ as two polyphase components of $H(z)$:

$$E_0(z) = \sum_{n=-\infty}^{\infty} h[2n]z^{-n},$$
$$E_1(z) = \sum_{n=-\infty}^{\infty} h[2n+1]z^{-n},$$

We have

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

- These representations hold whether $H(z)$ is FIR or IIR, causal or non-causal.
- The polyphase decomposition can be applied to any sequence, not just impulse response.

FIR and IIR Example

1. FIR filter: $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

   $\because H(z) = (1 + 3z^{-2}) + z^{-1}(2 + 4z^{-2}),$

   $\therefore E_0(z) = 1 + 3z^{-1}; \quad E_1(z) = 2 + 4z^{-1}$

2. IIR filter: $H(z) = \frac{1}{1-\alpha z^{-1}}$.

   Write into the form of $H(z) = E_0(z^2) + z^{-1}E_1(z^2)$:

   $$\because H(z) = \frac{1}{1-\alpha z^{-1}} \times \frac{1+\alpha z^{-1}}{1+\alpha z^{-1}} = \frac{1+\alpha z^{-1}}{1-\alpha^2 z^{-2}}$$
   $$= \frac{1}{1-\alpha^2 z^{-2}} + z^{-1}\frac{\alpha}{1-\alpha^{-2} z^{-2}}$$

   $\therefore E_0(z) = \frac{1}{1-\alpha^2 z^{-1}}; \quad E_1(z) = \frac{\alpha}{1-\alpha^{-2} z^{-1}}$

Extension to M Polyphase Components

For a given integer M and $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$, we have:

$$H(z) = \sum_{n=-\infty}^{\infty} h[nM]z^{-nM} + z^{-1} \sum_{n=-\infty}^{\infty} h[nM+1]z^{-nM} + \quad \dots \quad + z^{-(M-1)} \sum_{n=-\infty}^{\infty} h[nM+M-1]z^{-nM}$$

Type-1 Polyphase Representation

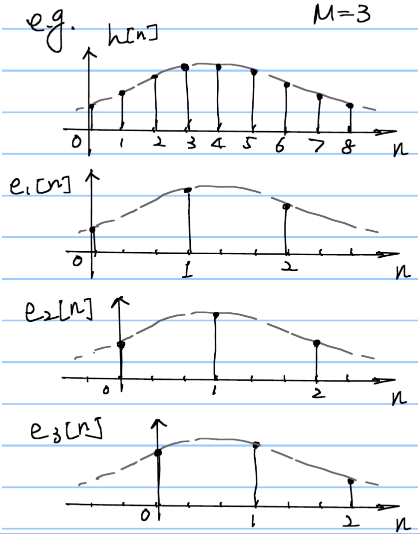
$$H(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_\ell(z^M)$$

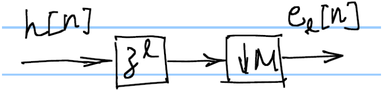
where the ℓ-th polyphase components of $H(z)$ given M is

$$E_\ell(z) \triangleq \sum_{n=-\infty}^{\infty} e_\ell[n]z^{-n} = \sum_{n=-\infty}^{\infty} h[nM+\ell]z^{-n}$$

Note: $0 \leq \ell \leq (M-1)$; strictly we may denote as $E_\ell^{(M)}(z)$.

Example: $M = 3$





z^{ℓ}: time advance

(there is a delay term when putting together the polyphase components)

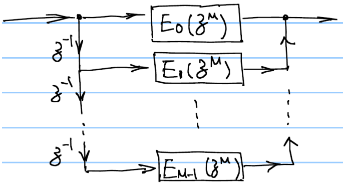
Alternative Polyphase Representation

If we define $R_\ell(z) = E_{M-1-\ell}(z),\ 0 \le \ell \le M-1$, we arrive at the

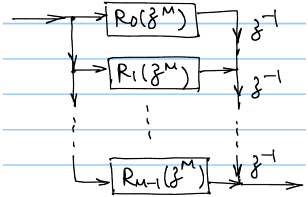
Type-2 polyphase representation

$$H(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_\ell(z^M)$$

Type-1: $E_k(z)$ is ordered consistently with the number of delays in the input



Type-2: reversely order the filter $R_k(z)$ with respect to the delays



Issues with Direct Implementation of Decimation Filters

Decimation Filters: 

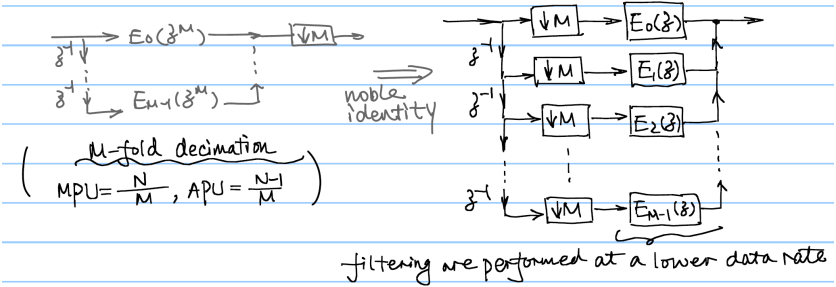
Question: Any wasteful effort in the direct implementation?

- The filtering is applied to all original signal samples, even though **only every** M **filtering output** is retained finally.
- Even if we let $H(z)$ operates only for time instants multiple of M and idle otherwise, all multipliers/adders have to produce results **within one step of time**.
- Can $\downarrow M$ be moved before $H(z)$?

  Only when $H(z)$ is a function of z^M, we can apply the noble identities to switch the order.

Efficient Structure for Decimation Filter

Apply Type-1 polyphase representation:



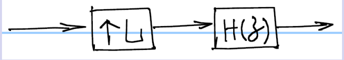
Computational Cost

For FIR filter $H(z)$ of length N:

- The total cost of N multipliers and $(N-1)$ adders is unchanged.
- Considering multiplications per input unit time (MPU) and additions per input unit time (APU),
  $E_k(z)$ now operates at a lower rate:
  only N/M **MPU and** $(N-1)/M$ **APU** are required.
- This is as opposed to N MPU and $(N-1)$ APU at every M instant of time and system idling at other instants, which leads to inefficient resource utilization.

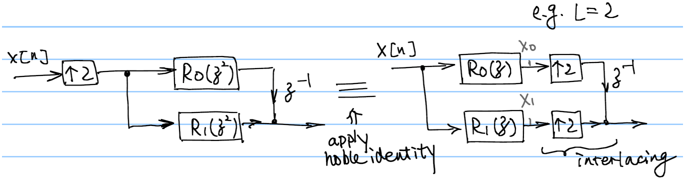
  (i.e., requires use fast additions and multiplications but use them only $1/M$ of time)

Polyphase for Interpolation Filters



Observe: the filter is applied to a signal at a high rate, even though many samples are zero when coming out of the expander.

Using the Type-2 polyphase decomposition:



$H(z) = z^{-1}R_0(z^2) + R_1(z^2)$:

- 2 polyphase components
- $R_k(z)$ is half length of $H(z)$

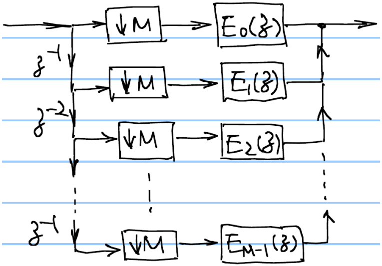
The complexity of the system is N MPU and $(N-2)$ APU.

General Cases

In general, for FIR filters with length N:

M-fold decimation:

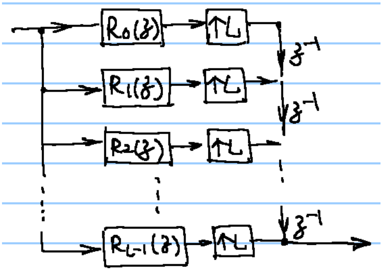
$$\text{MPU} = \frac{N}{M},\ \text{APU} = \frac{N-1}{M}$$



filtering is performed at a lower data rate

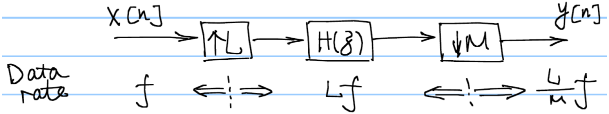
L-fold decimation:

$$\text{MPU} = N,\ \text{APU} = N - L$$



$$\text{APU} = (\frac{N}{L} - 1) \times L$$

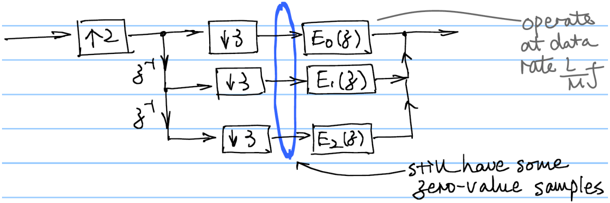
Fractional Rate Conversion



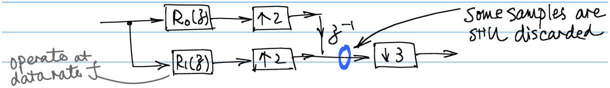
- Typically L and M should be chosen to have no common factors greater than 1 (o.w. it is wasteful as we make the rate higher than necessary only to reduce it down later)
- $H(z)$ filter needs to be fast as it operates in high data rate.
- The direct implementation of $H(z)$ is inefficient:
$$\begin{cases} \text{there are } L-1 \text{ zeros in between its input samples} \\ \text{only one out of } M \text{ samples is retained} \end{cases}$$

Example: $L = 2$ and $M = 3$

1. Use Type-1 polyphase decomposition (PD) for decimator:



2. Use Type-2 PD for interpolator:



Example: $L = 2$ and $M = 3$

3. Try to take advantage of both:

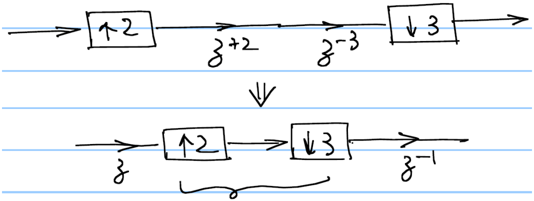
   Question: What's the lowest possible data rate to process? f/M

   Challenge: Can't move $\uparrow 2$ further to the right and $\downarrow 3$ to the left across the delay terms.

Trick to enable interchange of $\uparrow L$ and $\downarrow M$

$$z^{-1} = z^{-3} \cdot z^{2}$$

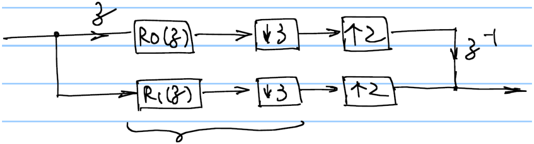
- z^{-3} and z^{2} can be considered as filters in z^{-M} and z^{+L}
- Noble identities can be applied:



can be interchanged as they are relatively prime

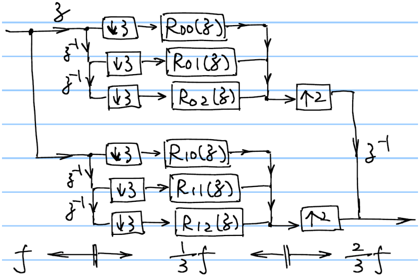
Overall Efficient Structure

Now it becomes



can move decimation earlier by Type-1 PD of $R_k(z)$

Finally,



$$R_0(z) = R_{00}(z^3) + z^{-1}R_{01}(z^3) + z^{-2}R_{02}(z^3)$$

$$R_1(z) = R_{10}(z^3) + z^{-1}R_{11}(z^3) + z^{-2}R_{12}(z^3)$$

Observations

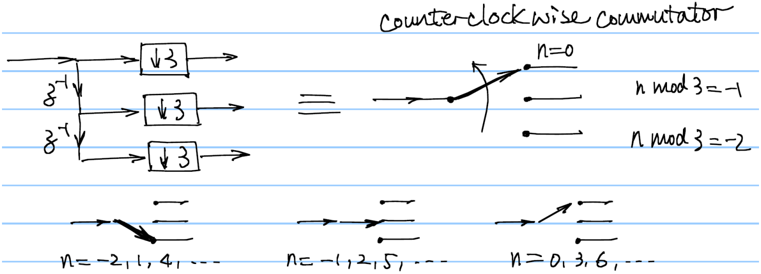
- For N-th order $H(z)$: MPU $= (N+1)/M \Rightarrow$ independent of L
- The final structure is the most efficient:
  $$\begin{cases} \text{Decimators are moved to the left of all computational units} \\ \text{Expanders are moved to the right of all computational units} \end{cases}$$
  Thus the computation is operated at the lowest possible rate.
- The above scheme works for arbitrary integers L and M as long as they are relatively prime.

  Under this condition, we have:
  1. $\exists n_0, n_1 \in \mathbb{Z}$ s.t. $n_1 M - n_0 L = 1$ (**Euclid's theorem**)
     We can then decompose $z^{-1} = z^{n_0 L} z^{-n_1 M}$
  2. $\uparrow L$ and $\downarrow M$ are interchangeable

Commutator Model: A Delay Chain followed by Decimators

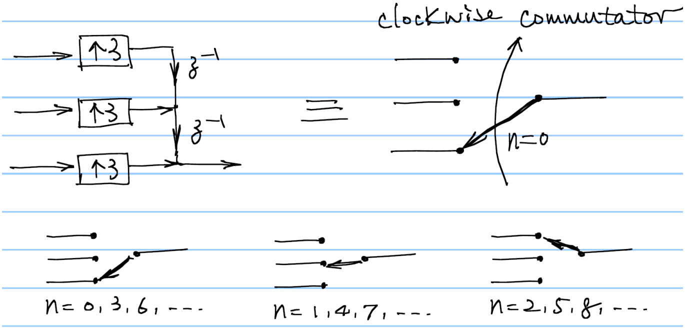
Polyphase implementation is often characterized by

1. A delay chain followed by a set of decimators,



Commutator Model: Expanders followed by A Delay Chain

2. A set of expanders followed by a delay chain



Commutator/switch model is an appealing conceptual tool to visualize these operations

Discussions: Linear Periodically Time Varying Systems

Some multirate systems that we have seen are linear periodically time varying (LPTV) systems.

e.g.,

for L=M

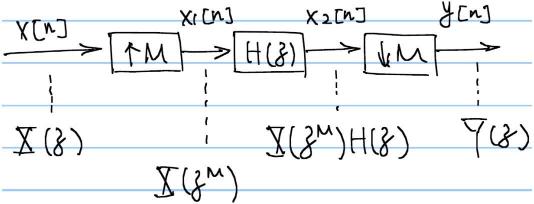
$$y[n] = \begin{cases} x[n] & \text{if } n \text{ is multiple of } M \\ 0 & \textit{otherwise} \end{cases}$$
$$= x[n] \cdot c[n]$$

$c[n]$ is a comb function: takes 1 for n is multiple of M and 0 o.w.

$\Rightarrow$ This is a linear system with periodically time varying response coefficients, and the period is M.

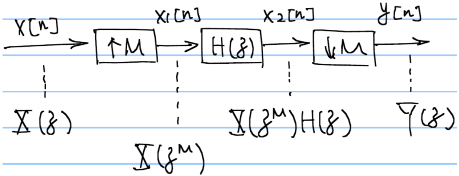
Time-invariant System with Decimator / Expander

Even though $\uparrow L$ and $\downarrow M$ are time-varying, a cascaded system having them as building blocks may become time-invariant.



This structure is the same as a fractional decimation system with $L = M$.

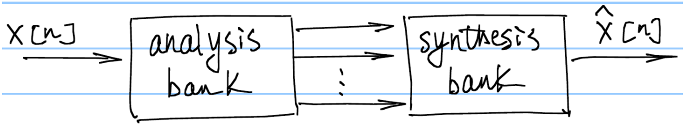
Time-invariant System with $\uparrow M$ & $\downarrow M$



details

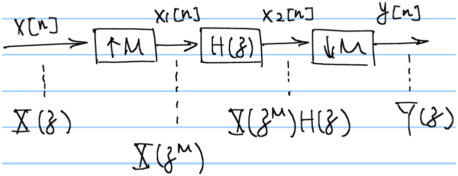
Recall: $[\mathbb{X}(z)]_{\downarrow M} = \frac{1}{M}\sum_{k=0}^{M-1} \mathbb{X}\left(W_M^k z^{1/M}\right)$

Perfect Reconstruction (PR) Systems



The above system is said to be a **perfect reconstruction** system if $\hat{x}[n] = cx[n - n_0]$ for some $c \neq 0$ and integer n_0, i.e., the output is identical to the input, except a constant multiplicative factor and some fixed delay.

Special Time-invariant System with $\uparrow M$ & $\downarrow M$

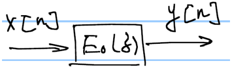


(back)

Recall: $[\mathbb{X}(z)]_{\downarrow M} = \frac{1}{M}\sum_{k=0}^{M-1}\mathbb{X}\left(W_M^k z^{1/M}\right)$

$$\mathbb{Y}(z) = \left[\mathbb{X}(z^M)H(z)\right]_{\downarrow M}$$
$$= \frac{1}{M}\sum_{k=0}^{M-1}\mathbb{X}\left(W_M^{Mk} z\right) H\left(W_M^k z^{1/M}\right) = \mathbb{X}(z)[H(z)]_{\downarrow M}$$

$[H(z)]_{\downarrow M}$ implies decimating the impulse response $h[n]$ by M-fold, corresponding to the 0-th polyphase component of $H(z)$.

$\Rightarrow \mathbb{Y}(z) = \mathbb{X}(z)E_0(z)$, i.e., 

, an LTI system.

Multi-rate Signal Processing
4. Multistage Implementations
5. Multirate Application: Subband Coding

Prof. Min Wu

University of Maryland, College Park

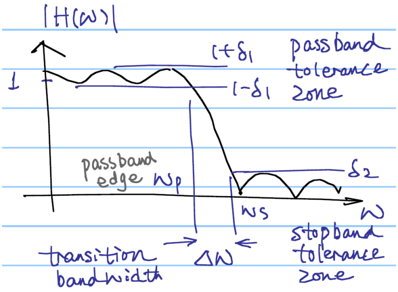
minwu@umd.edu

Updated: September 21, 2011

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu.

The LaTeX version includes contributions from Mr. Wei-Hong Chuang.

Preliminaries: Filter's magnitude response



Filter design theory

A linear phase FIR filter that satisfies this specification has order $N = g(\delta_1, \delta_2, \Delta W)$

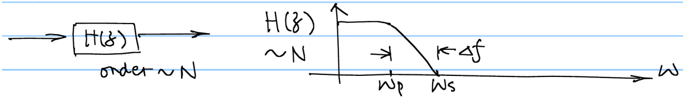
- as a function of δ_1, δ_2, and Δf
- $\Delta_f \approx \frac{\Delta W}{2\pi}$ (normalized transition b.w. $\in [0, 1]$)

For fixed ripple size, $N \propto \frac{1}{\Delta f}$:

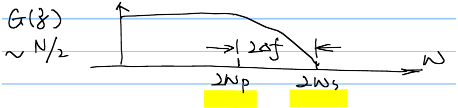
$\Delta f \uparrow \rightarrow N \downarrow$ (computation $\downarrow$)

Doubling Filter Transition Band

Consider an original LPF implementation



If we have a LPF with transition band $2\Delta f$, we may reduce the order by about half.



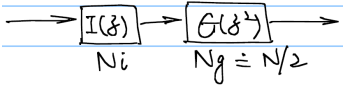
Double transition band leads to half of the required order for the filter.

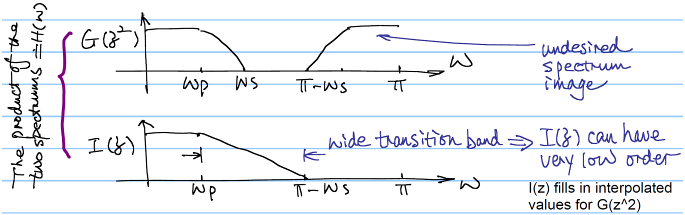
Interpolated FIR (IFIR)

Questions:

- With passband and stopband also doubled, what will be the response of a new filter that is an expanded version of the impulse response for $G(z)$, i.e., $G(z^2)$?
- What else is needed to get the same system response as $H(z)$?

New Interpolated FIR Design:





Multistage Decimation / Expansion

With what we have in IFIR design, reconsider now the efficient implementation of multirate filters:

e.g.,

$$\rightarrow \boxed{H(z)} \rightarrow \boxed{\downarrow 50} \rightarrow$$

$M = 50$

- Narrow passband for $H(z)$
  $\Rightarrow$ long filter needed
- Using polyphase representation
  $\Rightarrow$ need many decomposition components for large M!

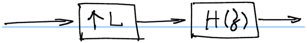
How about?

$$\rightarrow \boxed{H_1(z)} \rightarrow \boxed{\downarrow 25} \rightarrow \boxed{H_2(z)} \rightarrow \boxed{\downarrow 2} \rightarrow$$

Multistage implementation can be more efficient (in terms of computations per unit time).

Multistage Decimation / Expansion

Similarly, for interpolation,



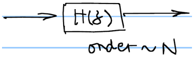
Summary

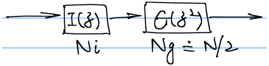
By implementing in multistage, not only the number of polyphase components reduces, but most importantly, the filter specification is less stringent and the overall order of the filters are reduced.

Exercises:

- Close book and think first how you would solve the problems.
- Sketch your solutions on your notebook.
- Then read V-book Sec. 4.4.

IFIR Design

Original system: 

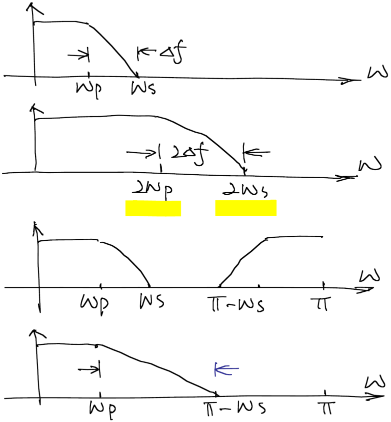
New system: 

$H(z) \sim N$

$G(z) \sim \frac{N}{2}$

$G(z^2)$

$I(z)$



(omit ripples in the sketches)

Doubled transition band leads to half of the required order for the filter

Note the undesired spectrum image

Wide transition band $\Rightarrow$ $I(z)$ can have very low order

$\divideontimes$ $G(2\omega) \times I(\omega) \approx H(\omega)$

Discussions

The complexity of the two-stage implementation is much less than that of the direct implementation.

- $G(z)$: the model filter
  (designed according to the "scaled" specification of $H(z)$)
- $I(z)$: image suppressor
- Number of adders: $N_i + N_g \ll N$
- Number of multipliers: $(N_i + 1) + (N_g + 1) \ll (N + 1)$

Principle of IFIR Design

$\Rightarrow$ Motivated multistage design from an efficient design technique of narrowband LPF known as IFIR.

- Applicable for designing any narrowband FIR filter (by itself not tied with $\uparrow L$ or $\downarrow M$)

Readings: Vaidyanathan's Book Sec. 4.4

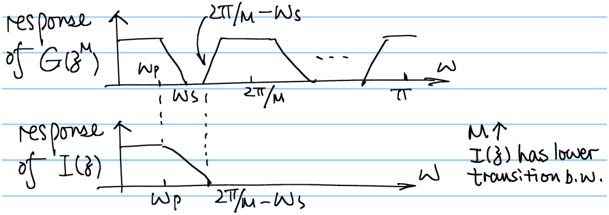
Extension to $M \geq 2$

In general, it is possible to stretch more, by an amount $M \geq 2$,

$$\longrightarrow \boxed{I(z)} \longrightarrow \boxed{G(z^M)} \longrightarrow$$

- so that the transition band of $G(z)$ can be even wider ($\approx M\Delta f$) and further reduces the order N_g
- Stopband edge in $G(z)$: $M\omega_s \leq \pi$
  $\Rightarrow M \leq \lfloor \frac{\pi}{\omega_s} \rfloor$

Extension to $M \geq 2$: Tradeoff

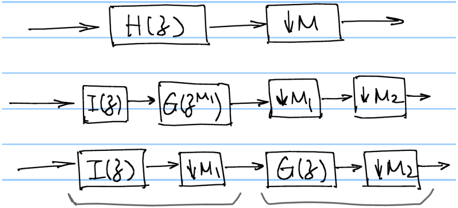


Tradeoff of the total cost: $M \uparrow$

- $G(z)$: transition b.w. $\uparrow \rightarrow$ order $\downarrow$
- $I(z)$: transition b.w. $\downarrow$ (could become very narrow) $\rightarrow$ order $\uparrow$

$\Rightarrow$ can search for optimal M.

Multistage Design of Decimation Filter



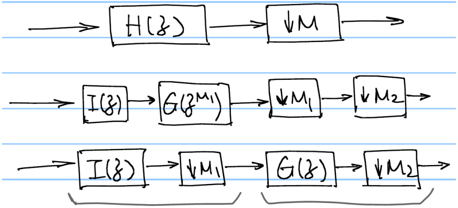
polyphase implementation each stage

$M = M_1 M_2$:

Choice of M_1 can be cast as an optimization problem

Rule of thumb: choose M_1 larger to reduce the computation complexity & data rate early on

Multistage Design Example: (1) Direct Design



polyphase implementation each stage

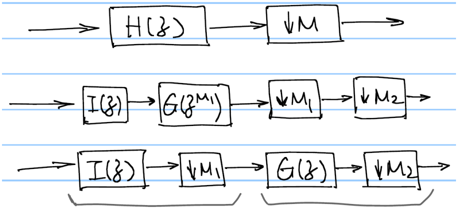
e.g., $M = 50$ fold decimation of an $8k$Hz signal

$H(z)$: $\delta_1 = 0.01$, $\delta_2 = 0.001$, passband edge $=$ 70Hz, stopband edge $=$ 80Hz

$\sim$ normalized $\Delta f = \frac{10}{8k} = \frac{1}{800}$

the order of direct equiripple filter design $\Rightarrow N = \mathbf{2028}$

Multistage Design Example: (2) Two-stage Design



polyphase implementation each stage

$M_1 = 25,\ M_2 = 2$

$G(z):\ \Delta f = 25 \times \frac{1}{800}$
$\omega_p = 0.4375\pi, \omega_s = 0.5\pi,$
$\delta_1 = 0.005, \delta_2 = 0.001$

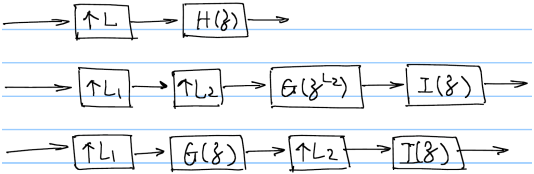
$\Rightarrow N_g = \mathbf{90}$

$I(z):\ \Delta f = 17 \times \frac{1}{800}$
$\omega_p = 0.0175\pi, \omega_s = 0.06\pi,$
$\delta_1 = 0.005, \delta_2 = 0.001$

$\Rightarrow N_i = \mathbf{139}$
higher order than $G(z)$ due to narrower transition

See spectrum sketch in Vaidyanathan's Book, Fig. 4.4-6.

Interpolation Filter



L_1 should be small to avoid too much increase in data rate and filter computation at early stage

e.g., $L = 50$: $L_1 = 2$, $L_2 = 25$

Summary

By implementing in multistage, not only the number of polyphase components reduces, but most importantly, the filter specification is less stringent and the overall order of the filters are reduced.

How to compress a signal?

- Tradeoff between bit rate and fidelity
- Many aspects to explore:
  use bits wisely; exploit redundancy; discard unimportant parts; ...
- Allocate bit rate strategically: equal allocation vs. focused effort

Compression Tool #1 (lossless if free from aliasing): Downsample a signal of limited bandwidth

(From what we learned about decimation in §1.1)

If a discrete-time signal is bandlimited with bandwidth smaller than 2π, the signal can be decimated by an appropriate factor without losing information.

- i.e., we don't need to keep that many samples
- Recall the example in §1.1.1: $|\omega| < \frac{2}{3}\pi$
  $\Rightarrow$ can change data rate to $\frac{2}{3}$ of original
- If signal spectrum support is in $(\omega_1, \omega_1 + \frac{2\pi}{M})$, we can decimate the signal by M fold without introducing aliasing.
  (Decimated signal may extend to entire 2π spectrum range)

Compression Tool #2 (lossy): Quantization

Dynamic range A of a signal: the value range

- Use a finite number of bits to represent a continuous valued sample via scalar quantization:

  partition A into N intervals, pick N representative values and use $\log_2 N$ bits to represent each value.

  $\rightarrow$ Simple quantization: uniform quantization

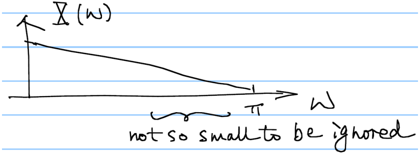
Compression Tool #2 (lossy): Quantization

- Quantify the "imprecision" between original and quantized:
  - maximum error $\max_x |x - \hat{x}|$
  - mean squared error $\mathbb{E}[(x - \hat{x})^2]$: easy to differential in an optimization formulation
- For a fixed amount of average error, signal with large dynamic range requires more bits in representation.
  e.g., uniform quantizer: max error $= A/(2N)$
  $\Rightarrow$ dynamic range $A \uparrow$ or # intervals $N \downarrow$ lead to higher error
- Non-uniform quantizer: may consider a few aspects
  1. keep relative error low (smaller stepsize in low value range)
  2. take account of signal's probability distribution and keep the expected error low (reduce error in most seen values)
     e.g., MMSE / Lloyd-Max quantizer

Non-bandlimited Signals

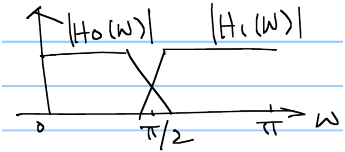
We often encounter signals that are not bandlimited, but have dominant frequency bands.

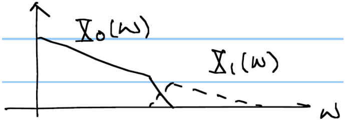
Question: How to use fewer bits to represent the signal and keep the imprecision low?



e.g. $x[n]$: 10kHz sampled signal, 16 bits/sample (to cover the dynamic range) ⇒ data bit rate 160$kbps$

Subband Coding





1. $x_0[n]$ and $x_1[n]$ are bandlimited and can be decimated
2. $\mathbb{X}_1(\omega)$ has smaller power s.t. $x_1[n]$ has smaller dynamic range, thus can be represented with fewer bits

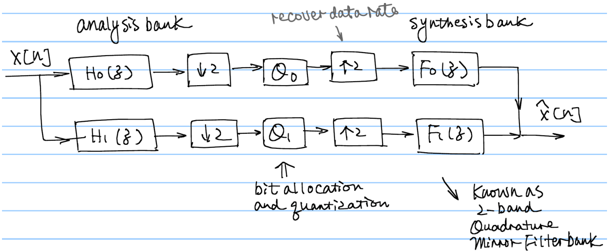
Suppose now to represent each subband signal, we need

$x_0[n]$: 16 bits / sample

$x_1[n]$: 8 bits / sample

$$\therefore 16 \times \frac{10k}{2} + 8 \times \frac{10k}{2} = 120k\text{bps}$$

Filter Bank for Subband Coding



Role of $F_k(z)$:

- eliminate spectrum images introduced by $\uparrow 2$
- If $\{H_k(z)\}$ is not perfect, the decimated subband signals may have aliasing.
- $\{F_k(z)\}$ should be chosen carefully so that the aliasing gets canceled at the synthesis stage (in $\hat{x}[n]$).

Applications in Digital Audio Systems

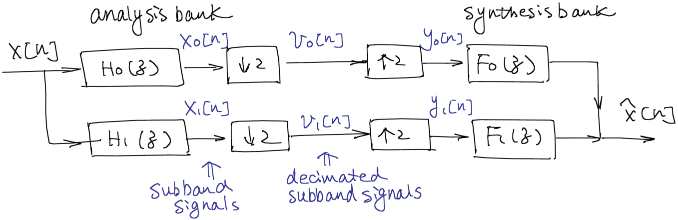
- During A/D conversion: Oversampling to alleviate the stringent requirements on the analog anti-aliasing filter
- During D/A conversion: Filter to remove spectrum images
- Fractional sampling rate conversion: Studio 48KHz vs. CD 44.1KHz

Readings to explore more: Vaidynathan Tutorial Sec. III-A.

Warm-up Exercise: Two-Channel Filter Bank

Under what conditions does a filter bank preserve information?

Derive the input-output relation in Z-domain.



Multi-rate Signal Processing
6. Quadrature Mirror Filter (QMF) Bank

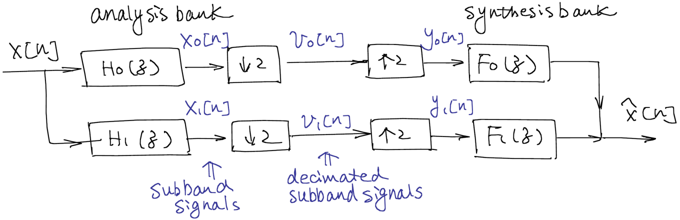
Electrical & Computer Engineering
University of Maryland, College Park

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu. The LaTeX slides were made by Prof. Min Wu and Mr. Wei-Hong Chuang.

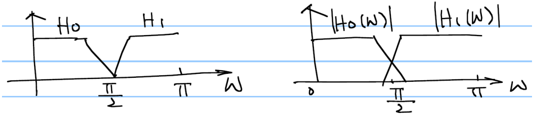
Contact: *minwu@umd.edu*. Updated: September 29, 2011.

Review: Two-channel Filter Bank

Recall: the 2-band QMF bank example in subband coding



Typical magnitude response



Overlapping filter response across $\pi/2$ may cause aliased subband signals

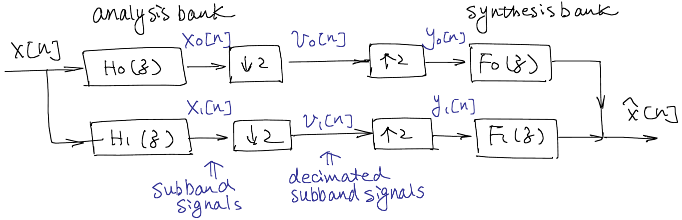
6.1 Errors Created in the QMF Bank

The reconstructed signal $\hat{x}[n]$ can differ from $x[n]$ due to

1. aliasing
2. amplitude distortion
3. phase distortion
4. processing of the decimated subband signal $v_k[n]$
   - quantization, coding, or other processing
   - inherent in practical implementation and/or depends on applications
     $\Rightarrow$ ignored in this section.

Readings: Vaidynathan Book 5.0-5.2; Tutorial Sec.VI.

Input-Output Relation



Examine the input-output relation: details

Input-Output Relation

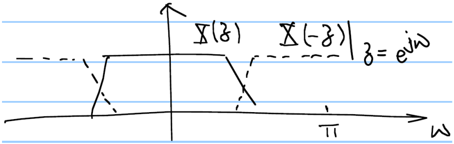
$$\begin{aligned}\hat{X}(z) &= \frac{1}{2}\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right]X(z) + \\ &\quad \frac{1}{2}\left[H_0(-z)F_0(z) + H_1(-z)F_1(z)\right]X(-z)\end{aligned}$$

In matrix-vector form: details

What is $X(-z)$?

- $X(-z)|_{z=e^{j\omega}} = X(\omega - \pi)$, i.e., shifted version of $X(\omega)$

  Referred to as the "alias term".



If $X(\omega)$ is not bandlimited by $\pi/2$, then $X(-z)$ may overlap with $X(z)$ spectrum.

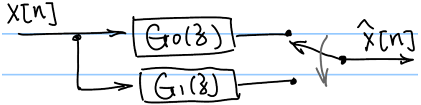
In the reconstructed signal $\hat{x}[n]$, this alias term reflects aliasing due to downsampling and residue imaging due to expansion.

Linear Periodically Time Varying (LPTV) Viewpoint

details Write $\hat{X}(z)$ expression as: $\hat{X}(z) = T(z)X(z) + A(z)X(-z)$

i.e., alternatingly taking output from one of the two LTI subsystems
(note: input and ouput have the same rate)

Linear Periodically Time Varying (LPTV) Viewpoint



If aliasing is cancelled (i.e., $A(z) = 0$), this will become LTI with transfer function $T(z)$.

Questions: Why we may want to permit some aliasing?

- To avoid excessive attenuation of input signal around $\omega = \frac{\pi}{2}$ and expensive $H_k(z)$ filters for sharp transition band, we permit some aliasing in the decimated analysis bank instead of trying to completely avoid it.
- We then choose synthesis filters so that the alias components in the two branches can cancel out each other.

Alias Cancellation

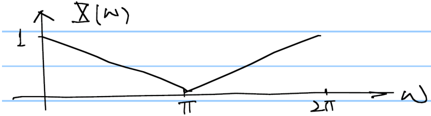
To cancel aliasing for all possible inputs $x[n]$ s.t.

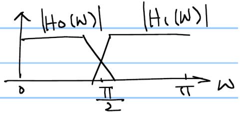
$$H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0,$$

we can choose

$$\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases} \qquad \text{(a sufficient condition)}$$

Example: sketch intermediate spectrums step-by-step

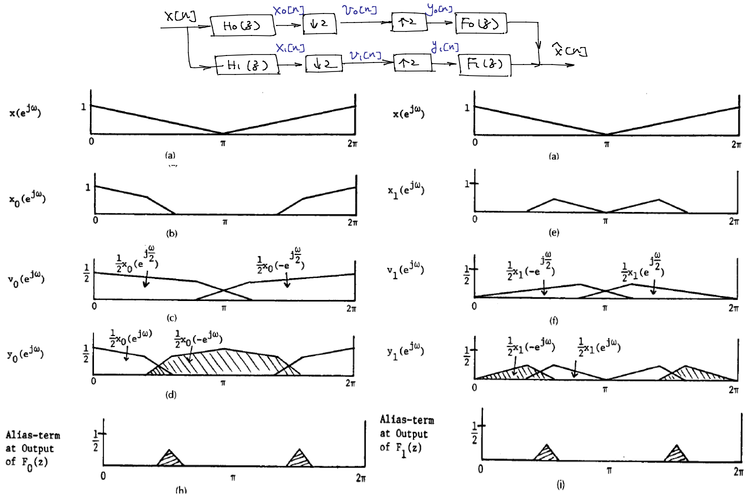




Alias Cancellation in the Spectrum

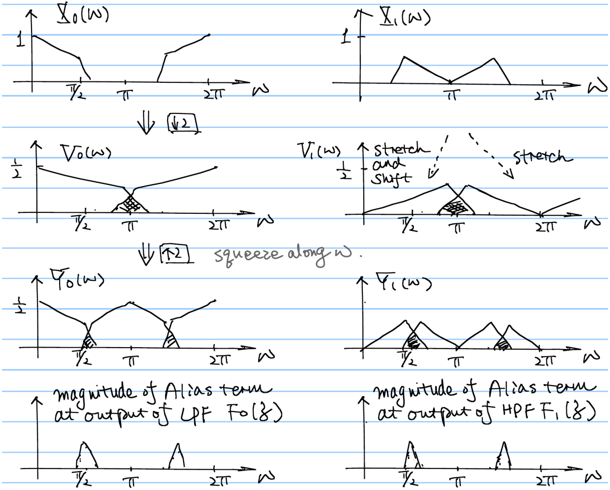
P.P. Vaidyanathan: "Multirate digital filters, filter banks, polyphase networks, andapplications: a tutorial", Proceedings of the IEEE, Jan 1990, Volume: 78, Issue: 1, pages 56-93. DOI: 10.1109/5.52200

Fig. 23. Illustration of various Fourier transforms in two-channel QMF bank. Here horizontal axis represents ω. (a) Typical input. (b) Transform. (c) Aliasing effect. (d) Imaging effect. (e) Using x_1. (f) Using v_1. (g) Using y_1. (h) Alias-term at output of $F_0(z)$. (i) Alias-term at output of $F_1(z)$.



Alias Cancellation in the Spectrum (sketch)

Assume $H_0(z)$ and $H_1(z)$ have some overlap and across $\pi/2$



possible to choose $F_k(z)$ to make these terms cancel each other out

Amplitude and Phase Distortions

Distortion Transfer Function

For an aliasing-free QMF bank, $\hat{X}(z) = T(z)X(z)$,

where $T(z) = \frac{1}{2}\left[H_0(z)F_0(z) + H_1(z)F_1(z)\right]$
$= \frac{1}{2}\left[H_0(z)H_1(-z) - H_1(z)H_0(-z)\right]$

This is called the distortion transfer function, or the overall transfer function of the alias-free system.

Let $T(\omega) = |T(\omega)|e^{j\phi(\omega)}$

To prevent amplitude distortion and phase distortion, $T(\omega)$ must be allpass (i.e. $|T(\omega)| = \alpha \neq 0$ for all ω, α is a constant) and linear phase (i.e., $\phi(\omega) = a + b\omega$ for constants a,b)

Properties of $T(z)$

- Perfect reconstruction (PR) property: if a QMF bank is free from aliasing, amplitude distortion and phase distortion, i.e., $T(z) = cz^{-n0} \Rightarrow \hat{x}[n] = cx[n - n_0]$

- With our above alias-free choice of $F_k(z)$, $T(z)$ is in the form of $T(z) = W(z) - W(-z)$, where $W(z) = H_0(z)H_1(-z)$.

  $\Rightarrow$ $T(z)$ has only odd power of z (as the even powers get cancelled), i.e., $T(z) = z^{-1}S(z^2)$ for some $S(z)$.

  So $|T(\omega)|$ has period of π (instead of 2π).
  And for real-coefficient filters, this implies $|T(\omega)|$ is symmetric w.r.t. $\pi/2$ for $0 \leq \omega < \pi$.

6.2 A Simple Alias-Free QMF System

Consider the analysis filters are related as

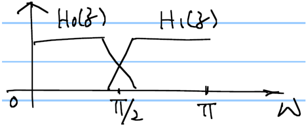
$$H_1(z) = H_0(-z)$$

For real filter coefficients, this means $|H_1(\omega)| = |H_0(\pi - \omega)|$.

$\because$ $|H_0(\omega)|$ symmetric w.r.t. $\omega = 0$; $|H_1(\omega)| \sim$ shift $|H_0(\omega)|$ by π.

i.e., $|H_1(\omega)|$ is a mirror image of $|H_0(\omega)|$ w.r.t. $\omega = \pi/2 = 2\pi/4$,
the "quadrature frequency" of the normalized sampling frequency.

If $H_0(z)$ is a good LPF,
then $H_1(z)$ is a good HPF.



(1) QMF Choice and Alias-free Condition

With QMF choice of $H_1(z) = H_0(-z)$, now the alias-free condition becomes

$$\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases} \quad \Rightarrow \quad \begin{cases} F_0(z) = H_0(z) \\ F_1(z) = -H_1(1z) \end{cases}$$

All four filters are completely determined by a single filter $H_0(z)$.

The distortion transfer function becomes

$$T(z) = \frac{1}{2}\left[H_0^2(z) - H_1^2(z)\right] = \frac{1}{2}\left[H_0^2(z) - H_0^2(-z)\right]$$

(2) Polyphase Representation of QMF

※ beneficial both computationally and conceptually

Let $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$ (Type-1 PD)

Then $H_1(z) = H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2)$

In matrix/vector form,

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix}$$

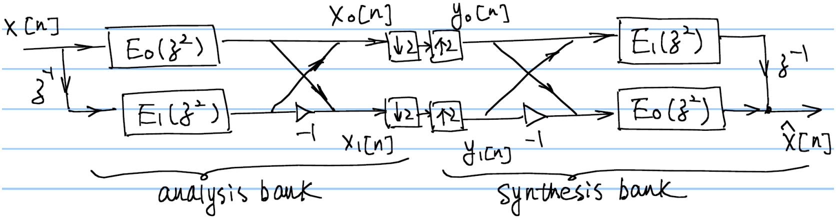
Similarly, for synthesis filters,

$$\begin{aligned} \begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} &= \begin{bmatrix} H_0(z) & -H_1(z) \end{bmatrix} \\ &= \begin{bmatrix} z^{-1}E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

Polyphase Representation: Signal Flow Diagram

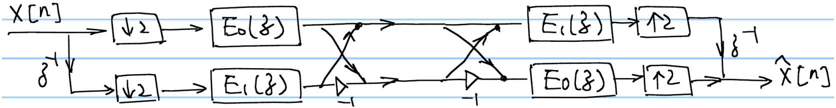
$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1}E_1(z^2) \end{bmatrix}$$

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1}E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



Polyphase Representation: Efficient Structure

Rearrange using nobel identities to obtain efficient implementation:



For $H_0(z)$ of length $N \quad \Rightarrow \quad E_k(z)$ has length $N/2$

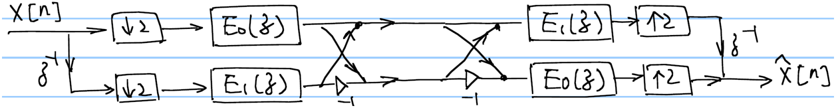
- Analysis bank: $N/2$ MPU, $N/2$ APU; same for synthesis bank
- Total: N MPU & APU

$\because H_0^2(z) = E_0^2(z^2) + E_1^2(z^2)z^{-2} + 2z^{-1}E_0(z^2)E_1^2(z^2)$

So the distortion transfer function becomes

$$T(z) = \frac{1}{2}\left[H_0^2(z) - H_0^2(-z)\right] = 2z^{-1}E_0(z^2)E_1(z^2)$$

Polyphase Representation: Matrix Form



In matrix form: (with MIMO transfer function for intermediate stages)

$$\underbrace{\begin{bmatrix} E_1(z) & 0 \\ 0 & E_0(z) \end{bmatrix}}_{\text{synthesis}} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \underbrace{\begin{bmatrix} E_0(z) & 0 \\ 0 & E_1(z) \end{bmatrix}}_{\text{analysis}}$$

$$= \begin{bmatrix} 2E_0(z)E_1(z) & 0 \\ 0 & 2E_0(z)E_1(z) \end{bmatrix}$$

※ Note: Multiplication is from left for each stage when intermediate signals are in column vector form.

Observations

The distortion transfer function of QMF

$T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$

- If $H_0(z)$ is FIR, so are $E_0(z), E_1(z)$ and $T(z)$.

- For $H_0(z)$ FIR and $H_1(z) = H_0(-z)$, the amplitude distortion can be eliminated **iff** $E_0(z)$ **and** $E_1(z)$ **represent a delay**:

$$\begin{cases} E_0(z) = c_0 z^{-n_0} \\ E_1(z) = c_1 z^{-n_1} \end{cases}$$

Observations

For $E_0(z)$ and $E_1(z)$ each representing a delay, we can only have analysis filters in the form of

$$\begin{cases} H_0(z) = c_0 z^{-2n_0} + c_1 z^{-(2n_1+1)} \\ H_1(z) = c_0 z^{-2n_0} - c_1 z^{-(2n_1+1)} \end{cases}$$

Such filters don't have sharp cutoff and good stopband attenuations.

Therefore $H_1(z) = H_0(-z)$ is not a good choice to build FIR perfect reconstruction QMF systems for such applications as subband coding.

(3) Eliminating Phase Distortions with FIR Filters

If $H_0(z)$ has linear phase, then we can show that

$$T(z) = \frac{1}{2}\left[H_0^2(z) - H_0^2(-z)\right]$$

also has linear phase (thus eliminating phase distortion).

Let $H_0(z) = \sum_{n=0}^{N} h_0[n] z^{-n}$ with $h_0[n]$ real. The linear phase and low pass conditions lead to $h_0[n] = h_0[N-n]$ (symmetric).

We can write $H_0(\omega) = e^{-j\omega\frac{N}{2}} \underbrace{R(\omega)}_{\text{real valued}}$

(3) Eliminating Phase Distortions with FIR Filters

$T(\omega)$ now becomes: details

Note: $|H_0(\omega)| = |R(\omega)|$ and $|H_0(\omega)|$ is even symmetric

$\Rightarrow T(\omega) = \frac{e^{-j\omega N}}{2}[|H_0(\omega)|^2 - (-1)^N |H_0(\pi - \omega)|^2]$

If N is even, $T(\omega)|_{\omega=\frac{\pi}{2}} = 0$, which brings severe amplitude distortion around $\omega = \pi/2$.

To avoid this, the filter order N should be **odd** (or length is even) so that $T(\omega) = \frac{e^{-j\omega N}}{2}\left[|H_0(\omega)|^2 + |H_0(\pi - \omega)|^2\right]$

(4) Minimizing Amplitude Distortion with FIR Filters

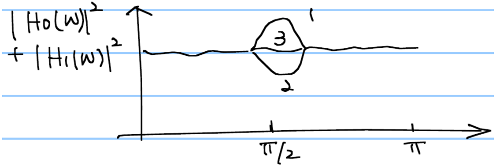
- Recall: after choosing $H_1(z) = H_0(-z)$, the amplitude distortion can be removed iff $H_0(z)$'s two polyphase components are pure delay.
  But such $H_0(z)$ doesn't have good low-pass response.

- For more flexible choices of $H_0(z)$ while eliminating aliasing and phase distortion, there will be some amplitude distortion.

- What we can do is to adjust the coefficients in $H_0(z)$ to minimize the amplitude distortion, i.e., to make $T(\omega)$ approximately constant:

$$|H_0(\omega)|^2 + |H_1(\omega)|^2 \approx 1$$

(4) Minimizing Amplitude Distortion with FIR Filters



1 – too much overlap b/w H_0 and H_1
2 – too little overlap
3 – good choice
(can be obtained by trial and error or by optimization formulation)

- Recall $T(z)$ has only odd power of z. For real-coeff. filter, $|T(\omega)|$ is symmetric w.r.t. $\pi/2$ for $0 \leq \omega < \pi$.
- By quadrature mirror condition, $|T(\omega)|$ is almost constant in the passbands of $H_0(z)$ and $H_1(z)$ if $H_0(z)$ has good passband and stopband responses.
- The main problem is with the transition band. The degree of overlap between $H_0(z)$ and $H_1(z)$ is crucial in determining this distortion.

See Vaidyanathan's Book §5.2.2 for details and examples

(5) Eliminating Amplitude Distortion with IIR Filters

How about IIR filters?

- The choice of $E_1(z) = \frac{1}{E_0(z)}$ can lead to perfect reconstruction and provide more room for designing $H(z)$.

  But the filters $H_k(z)$ would become IIR and may not provide desirable response.

- To completely eliminate amplitude distortion, $T(z)$ must be all-pass (which is IIR).

- Review: a 1st-order all-pass filter $G(z) = \frac{a^* + z^{-1}}{1 + az^{-1}}$

  $\Rightarrow |G(\omega)| = 1$; zero $= -1/a^*$, pole $= -a$ (conjugate reciprocal).

(5) Eliminating Amplitude Distortion with IIR Filters

One way to make $T(z)$ allpass is to choose $E_0(z)$ and $E_1(z)$ to be IIR and allpass.

Let $E_0(z) = \frac{a_0(z)}{2}$ and $E_1(z) = \frac{a_1(z)}{2}$ where $a_0(z)$ and $a_1(z)$ are allpass with $|a_0(\omega)| = |a_0(\omega)| = 1$.

The analysis filter becomes
$H_0(z) = E_0(z^2) + z^{-1}E_1(z^2) = \frac{a_0(z^2)+z^{-1}a_1(z^2)}{2}$

$\Rightarrow$ possible to have good $H(\omega)$ response with such all-pass polyphase form.
Explore PPV book 5.3

The overall distortion transfer function is allpass:
$T(z) = \frac{z^{-1}}{2} a_0(z^2)a_1(z^2)$

Phase Distortion with IIR Filters

- This design of QMF bank is free from amplitude distortion and aliasing, regardless of the details of the allpass filters $a_0(z)$ and $a_1(z)$.

- But the phase distortion remains due to the IIR components. The phase distortion is governed by the phase responses of $a_0(z)$ and $a_1(z)$.

Question: Can $a_0(z)$ and $a_1(z)$ be designed to cancel out phase distortion?

Note the difficulty in designing filters to meet many constraints.

Summary

Many “wishes” to consider toward achieving alias-free P.R. QMF:
(0) alias free, (1) phase distortion, (2) amplitude distortion,
(3) desirable filter responses.

Can't satisfy them all at the same time, so often meet most of them and try to approximate/optimize the rest.

A particular relation of synthesis-analysis filters to cancel alias:

$$\begin{cases} F_0(z) = H_1(-z) \\ F_1(z) = -H_0(-z) \end{cases} \quad \text{s.t. } H_0(-z)F_0(z) + H_1(-z)F_1(z) = 0.$$

We considered a specific relation between the analysis filters:
$H1(z) = H_0(-z)$ s.t. response symmetric w.r.t. $\omega = \pi/2$ (QMF)

With polyphase structure: $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$

Summary: $T(z) = 2z^{-1}E_0(z^2)E_1(z^2)$

Case-1 $H_0(z)$ **is FIR:**

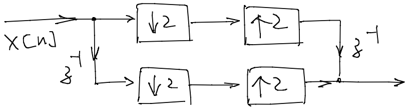
- P.R.: require polyphase components of $H_0(z)$ to be pure delay s.t. $H_0(z) = c_0 z^{-2n_0} + c_1 z^{-(2n_1+1)}$
  [cons] $H_0(\omega)$ response is very restricted.
- For more desirable filter response, the system may not be P.R., but can minimize distortion:
  - eliminate phase distortion: choose filter order N to be odd, and $h_0[n]$ be symmetric (linear phase)
  - minimize amplitude distortion: $|H_0(\omega)|^2 + |H_1(\omega)|^2 \approx 1$

Case-2 $H_0(z)$ **is IIR:**

- $E_1(z) = \frac{1}{E_0(z)}$ can get P.R. but restrict the filter responses.
- eliminate amplitude distortion: choose polyphase components to be all pass, s.t. $T(z)$ is all-pass, but may have some phase distortion

Look Ahead: Simple FIR P.R. Systems

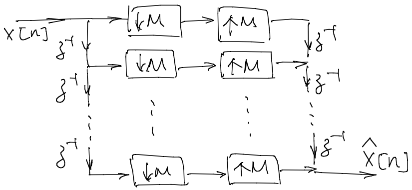
2-channel simple P.R. system:



How are $\hat{X}(z)$ and $X(z)$ related?

What are the equiv. $H_k(z)$ and $F_k(z)$?

Extend to M-channel:

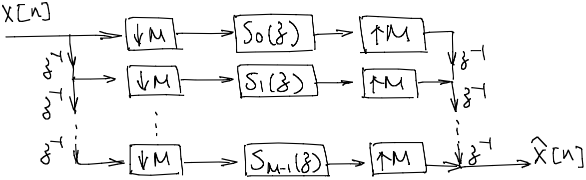


How are $\hat{X}(z)$ and $X(z)$ related?

What are the equiv. $H_k(z)$ and $F_k(z)$?

Interpretation: demultiplex then multiplex again

Look Ahead: Simple Filter Bank Systems

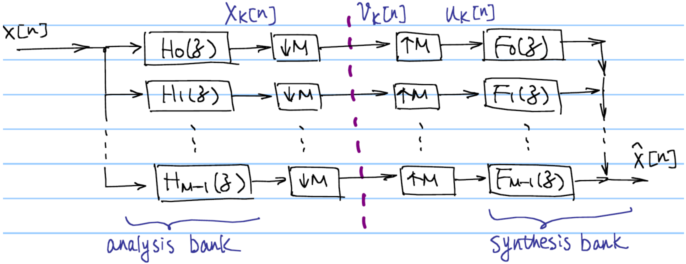


If all $S_k(z)$ are identical as $S(z)$, how are $\hat{X}(z)$ and $X(z)$ related?

How is this related to the simple M-channel P.R. system on the last page?

Look Ahead: *M*-channel filter bank

Study more general conditions of alias-free and PR; examine *M*-channel filter bank:



Derive the input-output relation. details

Input-Output Relation

Examine the input-output relation:

① Subband signals $X_k(z) = H_k(z) X(z)$ $k=0,1.$

② Decimated subband signals $V_k(z) = [X_k(z)]\big|_{\downarrow 2}$

$= \frac{1}{2} X_k(z^{1/2} W_2^0) + \frac{1}{2} X_k(z^{1/2} W_2^1)$

recall $W_M \triangleq e^{-j\frac{2\pi}{M}}$

$= \frac{1}{2} X_k(z^{1/2}) + \underbrace{\frac{1}{2} X_k(-z^{1/2})}$ $k=0,1.$

aliasing occurs if this part's spectrum overlaps with $X_k(z^{1/2})$

③ $Y_k(z) = V_k(z^2) = \frac{1}{2} X_k(z) + \frac{1}{2} X_k(-z)$

$= \frac{1}{2} H_k(z) X(z) + \frac{1}{2} H_k(-z) X(-z)$ $k=0,1.$

④ $\hat{X}(z) = F_0(z) Y_0(z) + F_1(z) Y_1(z)$

$= \frac{1}{2} [H_0(z) F_0(z) + H_1(z) F_1(z)] X(z)$

$+ \frac{1}{2} [H_0(-z) F_0(z) + H_1(-z) F_1(z)] \underbrace{X(-z)}$

Input-Output Relation

In matrix-vector form:

$$2\hat{X}(z) = [X(z) \quad X(-z)] \underbrace{\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}}_{\text{Define as } \mathcal{H}(z) \text{ "alias component matrix"}} \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}$$

$$= [F_0(z) \quad F_1(z)] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

LPTV (Linear Periodically Time Varying) Viewpoint

Write $\hat{X}(z)$ expression as :

$$\hat{X}(z) = T(z)X(z) + A(z)X(-z)$$

$$\Leftrightarrow \quad \hat{x}[n] = \sum_{k} \left(t[k] + (-1)^{n-k} a[k]\right) x[n-k]$$

Define $\begin{cases} g_0[k] = t[k] + (-1)^k a[k] \\ g_1[k] = t[k] - (-1)^k a[k] \end{cases}$

$$\Rightarrow \quad \hat{x}[n] = \begin{cases} g_0[n] * x[n] & n \text{ is even} \\ g_1[n] * x[n] & n \text{ is odd} \end{cases}$$

$$\begin{cases} X(z) = \sum_{n=-\infty}^{+\infty} x[n] z^{-n} \\ X(-z) = \sum_{n=-\infty}^{+\infty} x[n] \cdot (-1)^{-n} \cdot z^{-n} \end{cases}$$

i.e., alternatingly taking output from one of the two LTI subsystems
(note: input and ouput have the same rate)

Eliminating Phase Distortions with FIR Filters

$T(\omega)$ now becomes

$$\begin{aligned} T(\omega) &= \frac{1}{2}\left[H_0^2(\omega) - H_0^2(\omega-\pi) \right] \\ &= \frac{1}{2}\left[e^{-j\omega N} R^2(\omega) - e^{-j(\omega-\pi)N} R^2(\omega-\pi) \right] \\ &= \frac{e^{-j\omega N}}{2}\left[R^2(\omega) - (-1)^N R^2(\pi-\omega) \right] \\ &= \frac{e^{-j\omega N}}{2}\left[|H_0(\omega)|^2 - (-1)^N |H_0(\pi-\omega)|^2 \right] \end{aligned}$$

also used here $|H_0(\omega)| = |R(\omega)|$ and $|H_0(\omega)|$ being even symmetric

If N is even, $T(\omega)|_{\omega=\frac{\pi}{2}} = 0$, which brings severe amplitude distortion around $\omega = \pi/2$.

To avoid this, N should be odd so that $T(\omega) = \frac{e^{-j\omega N}}{2}\left[|H_0(\omega)|^2 + |H_0(\pi-\omega)|^2\right]$

Multi-rate Signal Processing
7. M-channel Maximally Decmiated Filter Banks

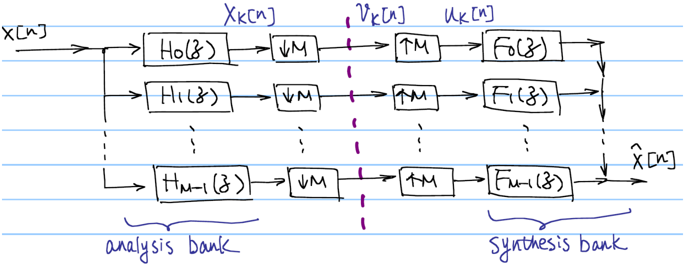
Electrical & Computer Engineering
University of Maryland, College Park

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu. The LaTeX slides were made by Prof. Min Wu and Mr. Wei-Hong Chuang.

Contact: *minwu@umd.edu*. Updated: October 6, 2011.

M-channel Maximally Decimated Filter Bank

To study more general conditions of alias-free and P.R., it becomes more convenient to examine M-channel filter bank.



As each of the filter has passband of about $2\pi/M$ wide, the subband signal output can be decimated up to M without substantial aliasing. The filter bank is said to be "maximally decimated" if this maximal decimation factor is used.

The Reconstructed Signal and Errors Created

[Readings: PPV Book 5.4, 5.5; Tutorial Sec. VIII]

Relations between $\hat{X}(z)$ and $X(z)$: (details)

$\hat{X}(z) = \sum_{l=0}^{M-1} A_\ell(z) X(W^\ell z)$

- $A_\ell(z) \triangleq \frac{1}{M} \sum_{k=0}^{M-1} H_k(W^\ell z) F_k(z)$, $0 \le \ell \le M-1$.
- $X(W^\ell z)|_{z=e^{j\omega}} = X(\omega - \frac{2\pi\ell}{M})$, i.e., shifted version from $X(\omega)$.
- $X(W^\ell z)$: ℓ-th aliasing term, $A_\ell(z)$: gain for this aliasing term.

Conditions for LPTV, LTI, and PR

- In general, the M-channel filter bank is a LPTV system with period M.

- The aliasing term can be eliminated for every possible input $x[n]$ iff $A_\ell(z) = 0$ for $1 \leq \ell \leq M-1$. When aliasing is eliminated, the filter bank becomes an LTI system:

$$\hat{X}(z) = T(z)X(z),$$

where $T(z) \triangleq A_0(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} H_k(z)F_k(z)$ is the overall transfer function, or distortion function.

- If $T(z) = cz^{-n_0}$, it is a perfect reconstruction system (i.e., free from aliasing, amplitude distortion, and phase distortion).

The Alias Component (AC) Matrix

From the definition of $A_\ell(z)$, we have in matrix-vector form:

$$M \underbrace{\begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix}}_{\mathcal{A}(z)} = \underbrace{\begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zW) & H_1(zW) & \cdots & H_{M-1}(zW) \\ \vdots & & & \\ H_0(zW^{M-1}) & H_1(zW^{M-1}) & \cdots & H_{M-1}(zW^{M-1}) \end{bmatrix}}_{\mathcal{H}(z)} \underbrace{\begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}}_{\mathbf{f}(z)}$$

$\mathcal{H}(z)$: $M \times M$ matrix called the "Alias Component matrix"

The condition for alias cancellation is

$$\mathcal{H}(z)\underline{\mathbb{f}}(z) = \underline{\mathbb{t}}(z), \quad \text{where } \underline{\mathbb{t}}(z) = \begin{bmatrix} MA_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The Alias Component (AC) Matrix

Now express the reconstructed signal as

$$\hat{X}(z) = \mathcal{A}^T(z)\mathcal{X}(z) = \frac{1}{M}\underline{\mathbb{f}}^T(z)\mathcal{H}^T(z)\mathcal{X}(z),$$

where $\mathcal{X}(z) = \begin{bmatrix} X(z) \\ X(zW) \\ \vdots \\ X(zW^{M-1}) \end{bmatrix}$.

Given a set of analysis filters $\{H_k(z)\}$, if $\det \mathcal{H}(z) \neq 0$, we can choose synthesis filters as $\underline{\mathbb{f}}(z) = \mathcal{H}^{-1}(z)\underline{\mathbb{t}}(z)$ to cancel aliasing and obtain P.R. by requiring

$$\underline{\mathbb{t}}(z) = \begin{bmatrix} cz^{-n_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Difficulty with the Matrix Inversion Approach

- $\mathcal{H}^{-1}(z)$ and thus the synthesis filters $\{F_k(z)\}$ can be IIR even if $\{H_k(z)\}$ are all FIR.
- Difficult to ensure $\{F_k(z)\}$ stability (all poles inside the unit circle)
- $\{F_k(z)\}$ may have high order even if the order of $\{H_k(z)\}$ is moderate
- ......

$\Rightarrow$ Take a different approach for P.R. design via polyphase representation.

Type-1 PD for $H_k(z)$

Using Type-1 PD for $H_k(z)$:

$$H_k(z) = \sum_{\ell=0}^{M-1} z^{-\ell} E_{k\ell}(z^M)$$

We have

$$\underbrace{\begin{bmatrix} H_0(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix}}_{\underline{\mathbb{h}}(z)} = \underbrace{\begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\ \vdots & & & \\ E_{M-1,0}(z^M) & \cdots & & E_{M-1,M-1}(z^M) \end{bmatrix}}_{\mathbb{E}(z^M)} \underbrace{\begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}}_{\underline{e}(z)}$$

$$\Longleftrightarrow \quad \underline{\mathbb{h}}(z) = \mathbb{E}(z^M)\,\underline{e}(z)$$

$\mathbb{E}(z^M)$: $M \times M$ Type-1 polyphase component matrix for analysis bank

Type-2 PD for $F_k(z)$

Similarly, using Type-2 PD for $F_k(z)$:

$$F_k(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_{\ell k}(z^M)$$

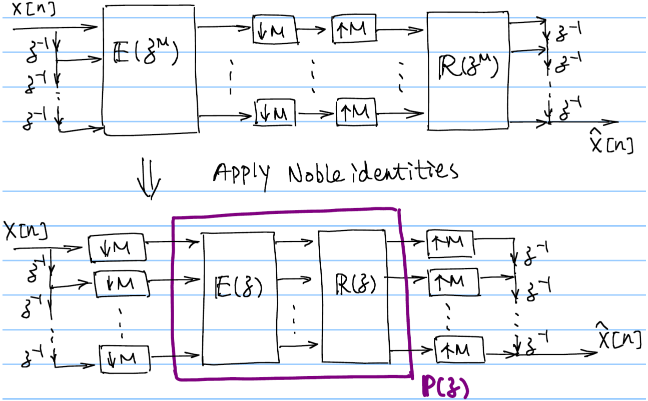
We have in matrix form:

$$\underbrace{\left[F_0(z) \ \cdots \ F_{M-1}(z)\right]}_{} = \underbrace{\left[z^{-(M-1)}, z^{-(M-2)}, \cdots, 1\right]}_{} \underbrace{\begin{bmatrix} R_{00}(z^M) & \cdots & R_{0,M-1}(z^M) \\ R_{10}(z^M) & \cdots & R_{1,M-1}(z^M) \\ \vdots & & \\ R_{M-1,0}(z^M) & \cdots & R_{M-1,M-1}(z^M) \end{bmatrix}}_{}$$

$$\Longleftrightarrow \underline{\mathbf{f}}^T(z) = \underline{\mathbf{e}}_B^T(z)\,\mathbb{R}(z^M)$$

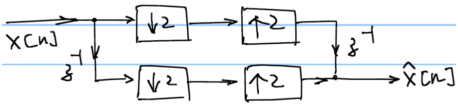
$\underline{\mathbb{e}}_B^T(z)$: reversely ordered version of $\underline{\mathbb{e}}(z)$
$\mathbb{R}(z^M)$: Type-2 polyphase component matrix for synthesis bank

Overall Polyphase Presentation



Combine polyphase matrices into one matrix: $\mathbb{P}(z) = \underbrace{\mathbb{R}(z)\mathbb{E}(z)}_{\text{note the order!}}$

Simple FIR P.R. Systems

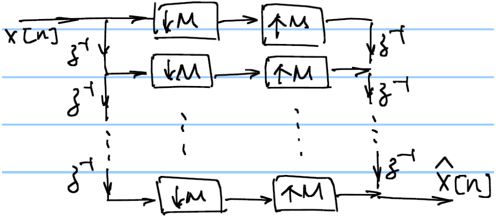


$\hat{X}(z) = z^{-1}X(z)$,
i.e., transfer function $T(z) = z^{-1}$

Extend to M channels:

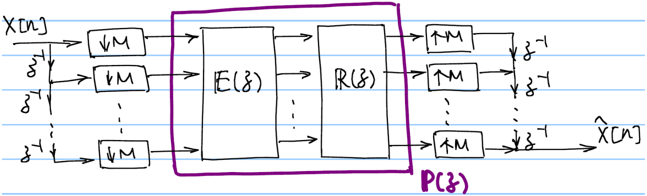
$H_k(z) = z^{-k}$
$F_k(z) = z^{-M+k+1}, 0 \leq k \leq M-1$
$\Rightarrow \hat{\mathbb{X}}(z) = z^{-(M-1)}\mathbb{X}(z)$

i.e. demultiplex then multiplex again



General P.R. Systems

Recall the polyphase implementation of M-channel filter bank:



Combine polyphase matrices into one matrix: $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z)$

If $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z) = \mathbb{I}$, then the system is equivalent to the simple system $\Rightarrow H_k(z) = z^{-k}$, $F_k(z) = z^{-M+k+1}$

In practice, we can allow $\mathbb{P}(z)$ to have some constant delay, i.e., $\mathbb{P}(z) = cz^{-m_0}\mathbb{I}$, thus $T(z) = cz^{-(Mm_0+M-1)}$.

Dealing with Matrix Inversion

To satisfy $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z) = \mathbb{I}$, it seems we have to do matrix inversion for getting the synthesis filters $\mathbb{R}(z) = (\mathbb{E}(z))^{-1}$.

Question: Does this get back to the same inversion problem we have with the viewpoint of the AC matrix $\underline{\mathbb{f}}(z) = \mathcal{H}^{-1}(z)\underline{\mathbb{t}}(z)$?

Solution:

- $\mathbb{E}(z)$ is a physical matrix that each entry can be controlled.
  In contrast, for $\mathcal{H}(z)$, only 1st row can be controlled (thus hard to ensure desired $H_k(z)$ responses **and** $\underline{\mathbb{f}}(z)$ stability)
- We can choose FIR $\mathbb{E}(z)$ s.t. $\det \mathbb{E}(z) = \alpha z^{-k}$ thus $\mathbb{R}(z)$ can be FIR (and has determinant of similar form).

Summary: With polyphase representation, we can choose $\mathbb{E}(z)$ to produce desired $H_k(z)$ and lead to simple $\mathbb{R}(z)$ s.t. $\mathbb{P}(z) = cz^{-k}\mathbb{I}$.

Paraunitary

A more general way to simplify the need of matrix inversion:

Constrain $\mathbb{E}(z)$ to be **paraunitary**: $\tilde{\mathbb{E}}(z)\mathbb{E}(z) = d\mathbb{I}$

Here $\tilde{\mathbb{E}}(z) = \mathbb{E}_*^T(z^{-1})$, i.e. taking conjugate of the transfer function coeff., replace z with z^{-1} that corresponds to time reversely order the filter coeff., and transpose.

For further exploration: PPV Book Chapter 6.

Relation b/w Polyphase Matrix $\mathbb{E}(z)$ and AC Matrix $\mathcal{H}(z)$

The relation between $\mathbb{E}(z)$ and $\mathcal{H}(z)$ can be shown as:

$$\mathcal{H}(z) = [\mathbb{W}^*]^T \, \mathbb{D}(z) \qquad \mathbb{E}^T(z^M)$$

where $\mathbb{W}$ is the $M \times M$ DFT matrix, and a diagonal delay matrix

$$\mathbb{D}(z) = \begin{bmatrix} 1 & & & \\ & z^{-1} & & \\ & & \ddots & \\ & & & z^{-(M-1)} \end{bmatrix}$$

(details) See also the homework.

Detailed Derivations

The Reconstructed Signal and Errors Created

① $X_k(z) = H_k(z) X(z)$

② $V_k(z) = \frac{1}{M} \sum_{\ell=0}^{M-1} \underbrace{H_k(W_M^\ell z^{1/M}) X(W_M^\ell z^{1/M})}_{X_k(W_M^\ell z^{1/M})}$

where $W_M = e^{-j2\pi/M}$

③ $U_k(z) = V_k(z^M) = \frac{1}{M} \sum_{\ell=0}^{M-1} H_k(W_M^\ell z) X(W_M^\ell z)$

④ $\hat{X}(z) = \sum_{k=0}^{M-1} F_k(z) U_k(z)$

$= \sum_{\ell=0}^{M-1} \underbrace{\left[\frac{1}{M} \sum_{k=0}^{M-1} H_k(W^\ell z) F_k(z)\right]} X(W^\ell z)$

$= \sum_{\ell=0}^{M-1} A_\ell(z) X(W^\ell z)$

- $A_\ell(z) \triangleq \frac{1}{M} \sum_{k=0}^{M-1} H_k(W^\ell z) F_k(z)$, $0 \leq \ell \leq M-1$.
- $\mathbb{X}(W^\ell z)|_{z=e^{j\omega}} = \mathbb{X}(\omega - \frac{2\pi\ell}{M})$, i.e., shifted version from $\mathbb{X}(\omega)$.
- $\mathbb{X}(W^\ell z)$: ℓ-th aliasing term, $A_\ell(z)$: gain for this aliasing term.

Review: Matrix Inversion

$\mathcal{H}(z)^{-1} = \frac{Adj[\mathcal{H}(z)]}{\det[\mathcal{H}(z)]}$

Adjugate or classical adjoint of a matrix:

$\{Adj[\mathcal{H}(z)]\}_{ij} = (-1)^{i+j} M_{ji}$

where M_{ji} is the (j, i) minor of $\mathcal{H}(z)$ defined as the determinant of the matrix by deleting the j-th row and i-th column.

An Example of P.R. Systems

$$H_0(z) = 2 + z^{-1},\ \ H_1(z) = 3 + 2z^{-1},$$

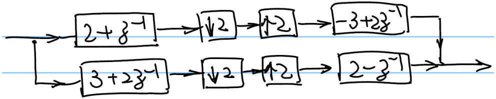
$$\mathbb{E}(z) = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix},\ \mathbb{E}^{-1}(z) = \frac{\text{Adj } \mathbb{E}(z)}{\det \mathbb{E}(z)} = 1 \times \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$$

Choose $\mathbb{R}(z) = \mathbb{E}^{-1}(z)$ s.t. $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z) = \mathbb{I}$,

$$\therefore \mathbb{R}(z) = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 1 \end{bmatrix} \mathbb{R}(z^2) = \begin{bmatrix} 2z^{-1} - 3, & -z^{-1} + 2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} F_0(z) = -3 + 2z^{-1} \\ F_1(z) = 2 - z^{-1} \end{cases}$$



Relation b/w Polyphase Matrix $\mathbb{E}(z)$ and AC Matrix $\mathcal{H}(z)$

From the definition of $\mathcal{H}(z)$ and $\underline{\mathbb{h}}(z)$, we have

$$\begin{aligned}\mathcal{H}^T(z) &= \left[\; \underline{\mathbb{h}}(z) \quad \underline{\mathbb{h}}(zW) \quad \cdots \quad \underline{\mathbb{h}}(zW^{M-1}) \;\right] \\ &= \mathbb{E}(z^M)\left[\; \underline{\mathbb{e}}(z) \quad \underline{\mathbb{e}}(zW) \quad \cdots \quad \underline{\mathbb{e}}(zW^{M-1}) \;\right]\end{aligned}$$

Examine $\underline{\mathbb{e}}(zW^k)$, $k = 0, 1, \ldots, (M-1)$:

$$\underline{\mathbb{e}}(zW^k) = \begin{bmatrix} 1 \\ (zW^k)^{-1} \\ \cdots \\ (zW^k)^{-(M-1)} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & \\ & z^{-1} & & \\ & & \ddots & \\ & & & z^{-(M-1)} \end{bmatrix}}_{\text{define as } \mathbb{D}(z)\text{,a diagonal delay matrix}} \begin{bmatrix} 1 \\ W^{-k} \\ \cdots \\ W^{-(M-1)k} \end{bmatrix}$$

Relation b/w Polyphase Matrix $\mathbb{E}(z)$ and AC Matrix $\mathcal{H}(z)$

Put together: $\mathcal{H}^T(z) = \mathbb{E}(z^M)\mathbb{D}(z)\mathbb{W}^*$

Thus we arrive at the relation between $\mathbb{E}(z)$ and $\mathcal{H}(z)$:

$$\mathcal{H}(z) = [\mathbb{W}^*]^T \, \mathbb{D}(z)\mathbb{E}^T(z^M)$$

Note: $[\mathbb{W}^*]^T$ is equal to $\mathbb{W}^*$ due to symmetry of $M \times M$ DFT matrix $\mathbb{W}$

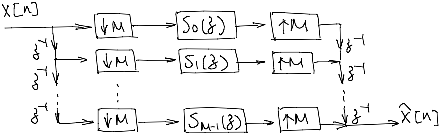
Multi-rate Signal Processing
8. General Alias-Free Conditions for Filter Banks
9. Tree Structured Filter Banks and Multiresolution Analysis

Electrical & Computer Engineering
University of Maryland, College Park

Acknowledgment: ENEE630 slides were based on class notes developed by Profs. K.J. Ray Liu and Min Wu. The LaTeX slides were made by Prof. Min Wu and Mr. Wei-Hong Chuang.

Contact: *minwu@umd.edu*. Updated: October 6, 2011.

Recall: Simple Filter Bank Systems

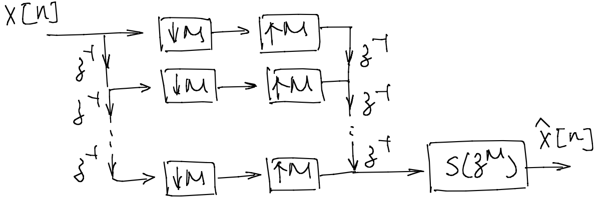


If all $S_k(z)$ are identical as $S(z)$:

$\mathbb{P}(z) = S(z)\mathbb{I}$

$\Rightarrow \hat{X}(z) = z^{-(M-1)}S(z^M)X(z)$

Alias Free



General Alias-free Condition

Recall from Section 7: The condition for alias cancellation in terms of $\mathcal{H}(z)$ and $\underline{\mathbb{f}}(z)$ is

$$\mathcal{H}(z)\underline{\mathbb{f}}(z) = \underline{\mathbb{t}}(z) = \begin{bmatrix} MA_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Theorem

A M-channel maximally decimated filter bank is alias-free <u>iff</u> the matrix $\mathbb{P}(z) = \mathbb{R}(z)\mathbb{E}(z)$ is **pseudo circulant**.

[Readings: PPV Book 5.7]

Circulant and Pseudo Circulant Matrix

(right-)circulant matrix

$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ P_2(z) & P_0(z) & P_1(z) \\ P_1(z) & P_2(z) & P_0(z) \end{bmatrix}$$

Each row is the right circular shift of previous row.

pseudo circulant matrix

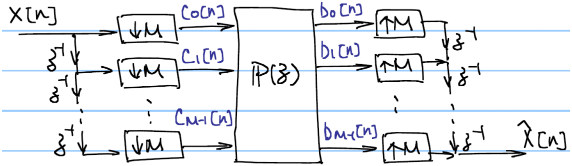
$$\begin{bmatrix} P_0(z) & P_1(z) & P_2(z) \\ z^{-1}P_2(z) & P_0(z) & P_1(z) \\ z^{-1}P_1(z) & z^{-1}P_2(z) & P_0(z) \end{bmatrix}$$

Adding z^{-1} to elements below the diagonal line of the circulant matrix.

- Both types of matrices are determined by the 1st row.
- Properties of pseudo circulant matrix (or as an alternative definition): Each column as up-shift version of its right column with z^{-1} to the wrapped entry.

Proof of the Theorem

(Details)



Denote $\mathbb{P}(z) = [P_{s,\ell}(z)]$.

Overall Transfer Function

The overall transfer function $T(z)$ after aliasing cancellation:

$\hat{X}(z) = T(z)X(z)$, where

$$T(z) = z^{-(M-1)}\{P_{0,0}(z^M) + z^{-1}P_{0,1}(z^M) + \cdots + z^{-(M-1)}P_{0,M-1}(z^M)\}$$

(Details)

Most General P.R. Conditions

Necessary and Sufficient P.R. Conditions

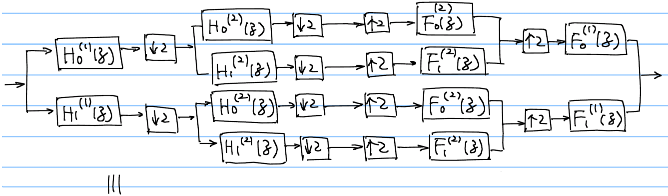
$$\mathbb{P}(z) = cz^{-m_0} \begin{bmatrix} 0 & \mathbb{I}_{M-r} \\ z^{-1}\mathbb{I}_r & 0 \end{bmatrix} \text{ for some } r \in 0, ..., M-1.$$

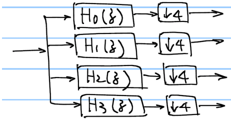
When $r = 0$, $\mathbb{P}(z) = \mathbb{I} \cdot cz^{-m_0}$, as the sufficient condition seen in §1.7.3.

(Details)

(Binary) Tree-Structured Filter Bank

A multi-stage way to build M-channel filter bank:
Split a signal into 2 subbands $\Rightarrow$ further split one or both subband signals into 2 $\Rightarrow \cdots$

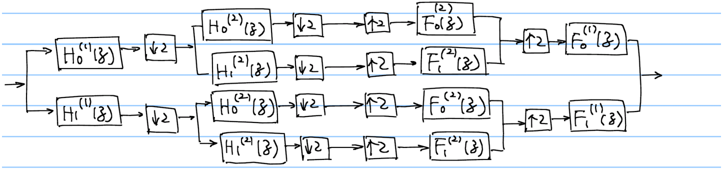




where
$H_0(z) = H_0^{(1)}(z)\, H_0^{(2)}(z^2)$ (by noble identities)
- - -

Question: Under what conditions is the overall system free from aliasing? How about P.R.?

(Binary) Tree-Structured Filter Bank

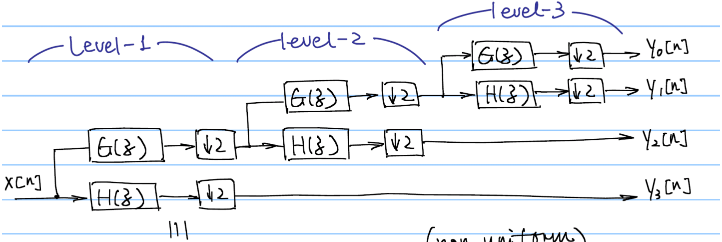


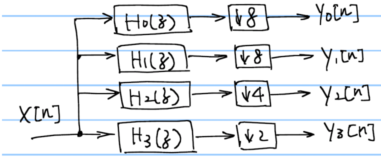
- Can analyze the equivalent filters by noble identities.
- If a 2-channel QMF bank with $H_0^{(K)}(z)$, $H_1^{(K)}(z)$, $F_0^{(K)}(z)$, $F_1^{(K)}(z)$ is alias-free, the complete system above is also alias-free.
- If the 2-channel system has P.R., so does the complete system.

[Readings: PPV Book 5.8]

Multi-resolution Analysis: Analysis Bank

Consider the variation of the tree structured filter bank (i.e., only split one subband signals)

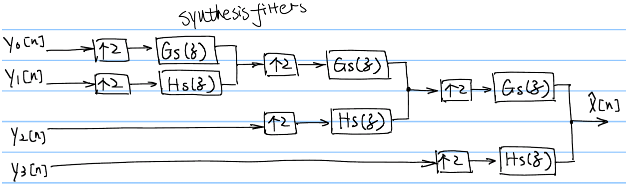




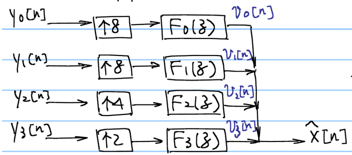
(non-uniform) Analysis Bank

* Typically $[G(z), H(z)]$ is a lowpass/highpass pair with cutoff $\approx \pi/2$ (as in 2-ch. QMF).

Multi-resolution Analysis: Synthesis Bank



$\equiv$



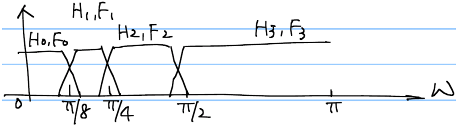
Synthesis Bank

* $\{v_k[n]\}$ have the same sampling rate as $x[n]$ and $\hat{x}[n]$

They are called the multiresolution components.

Discussions

(1) The typical frequency response of the equivalent analysis and synthesis filters are:



(2) The multiresolution components $v_k[n]$ at the output of $F_k(z)$:

- $v_0[n]$ is a lowpass version of $x[n]$ or a "coarse" approximation;
- $v_1[n]$ adds some high frequency details so that $v_0[n] + v_1[n]$ is a finer approximation of $x[n]$;
- $v_3[n]$ adds the finest ultimate details.

Discussions

(3) If 2-ch QMF with $G(z)$, $F(z)$, $G_s(z)$, $F_s(z)$ has P.R. with unit-gain and zero-delay, we have $x[n] = x[n]$.

(4) For compression applications: can assign more bits to represent the coarse info, and the remaining bits (if available) to finer details by quantizing the refinement signals accordingly.

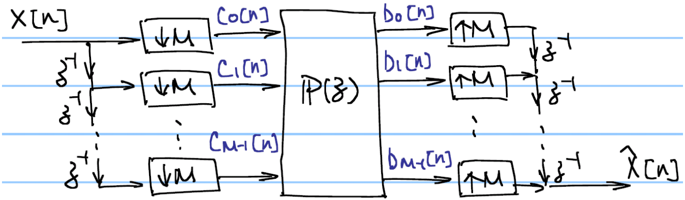
Brief Note on Subband vs Wavelet Coding

- The **octave (dyadic)** frequency partition can reflect the **logarithmic** characteristics in human perception.
- Wavelet coding and subband coding have many similarities (e.g. from filter bank perspectives)
  - Traditionally subband coding uses filters that have little overlap to isolate different bands
  - Wavelet transform imposes smoothness conditions on the filters that usually represent a set of basis generated by shifting and scaling (dilation) of a mother wavelet function
  - Wavelet can be motivated from overcoming the poor time-domain localization of short-time FT

$\Rightarrow$ Explore more in Proj#1. See PPV Book Chapter 11

Detailed Derivations

Details of the Proof (1)



Denote $\mathbb{P}(z) = [P_{s,\ell}(z)]$.

Details of the Proof (2)

①

Taking ZT on $c_\ell[n]$, we have:

$$C_\ell(z) = \frac{1}{M}\sum_{k=0}^{M-1} \left(W_M^k z^{1/M}\right)^{-\ell} X\left(W_M^k z^{1/M}\right) \quad \text{for } 0 \le \ell \le M-1$$

$$W_M = e^{-j2\pi/M}.$$

For ZT on $b_\ell[n]$, we have:

$$B_s(z) = \sum_{\ell=0}^{M-1} P_{s,\ell}(z)\, C_\ell(z)$$

$$\hat{X}(z) = \sum_{s=0}^{M-1} z^{-(M-s-1)} B_s(z^M) = \sum_{s=0}^{M-1}\sum_{\ell=0}^{M-1} z^{-(M-s-1)} P_{s,\ell}(z^M)\, C_\ell(z^M)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X(W_M^k z) \sum_{\ell=0}^{M-1} W_M^{-k\ell} \underbrace{\sum_{s=0}^{M-1} z^{-\ell} z^{-(M-s-1)} P_{s,\ell}(z^M)}_{\triangleq V_\ell(z)}$$

We have alias-free iff $\sum_{\ell=0}^{M-1} W_M^{-k\ell} V_\ell(z) = 0$ for $k \ne 0$.

Details of the Proof (3)

② The above nece.-suff. condition in matrix-vector form is:

$$\mathbf{W}^H \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $*$ represents a non-zero entry, A^H represents the transpose-conjugate of a matrix A, and $\mathbf{W}$ is the $M \times M$ DFT matrix:

$[\mathbf{W}]_{km} = W_M^{km} = e^{-j2\pi \frac{km}{M}}$, $0 \le k \le M-1$, $0 \le m \le M-1$, $W^H W = W W^H = M I$.

$$\therefore \begin{bmatrix} V_0(z) \\ \vdots \\ V_{M-1}(z) \end{bmatrix} = \mathbf{W} \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{W}\text{'s first column multiplied by } * = \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix}$$

all-one entries

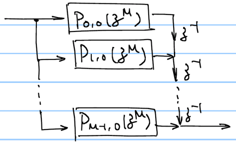
i.e. $V_0(z) = V_1(z) = \cdots = V_{M-1}(z) \triangleq V(z)$

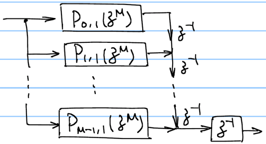
Details of the Proof (4)

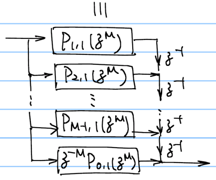
③ From the definition $V_\ell(z) = \sum_{s=0}^{M-1} z^{-\ell} z^{-(M-1-s)} P_{s,\ell}(z^M)$,

we have:

$$V_0(z) = \sum_{s=0}^{M-1} z^{-(M-1-s)} P_{s,0}(z^M) \qquad V_1(z) = z^{-1} \sum_{s=0}^{M-1} z^{-(M-1-s)} P_{s,1}(z^M)$$







Since $V_0(z) = V_1(z)$, we have

$$\begin{bmatrix} P_{0,0}(z) \\ P_{1,0}(z) \\ \vdots \\ P_{M-1,0}(z) \end{bmatrix} = \begin{bmatrix} P_{1,1}(z) \\ P_{2,1}(z) \\ \vdots \\ z^{-1}P_{0,1}(z) \end{bmatrix}$$

i.e. each column is the up-shift version of its right column with z^{-1} to the wraped entry

$$\begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix}$$

identical along diagonal directions

Concluding the Proof

The same relation can be found for other columns.

Thus $\mathbb{P}(z)$ is pseudo circulant for an alias-free system as a necessary and sufficient condition.

We've done with the proof of the theorem.

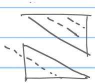
Overall Transfer Function

The overall transfer function $T(z)$ after aliasing cancellation:

$\hat{X}(z) = T(z)X(z)$, where
$T(z) = \frac{1}{M}\sum_{\ell=0}^{M-1} V_\ell(z) = \sum_{\ell=0}^{M-1}\sum_{s=0}^{M-1} z^{-(M-1+\ell-s)} P_{s,\ell}(z^M)$

We can represent entries of pseudo-circulant matrix in terms of the 0-th row entries:

Since $\begin{cases} P_{s,\ell}(z) = P_{0,\ell-s}(z) & \text{for } \ell \geq s, \text{ and} \\ P_{s,\ell}(z) = z^{-1} P_{0,M+\ell-s}(z) & \text{for } \ell < s; \end{cases}$



Let $r \triangleq \ell - s$, we have

$$z^{-(\ell-s)} P_{s,\ell}(z^M) = \begin{cases} z^{-r} P_{0,r}(z^M) & \text{for } 0 \leq r \leq M-1 \\ z^{-(M+r)} P_{0,M+r}(z^M) & \text{for } r<0 \text{ such that } 0 < M+r \leq M-1 \end{cases}$$

∘ for each $P_{0,r}(z^M)$, $0 \leq r \leq M-1$, we have M such terms in $T(z)$]

Overall Transfer Function

$$\Rightarrow T(z) = \frac{1}{M} z^{-(M-1)} \sum_{r=0}^{M-1} M \times z^{-r} P_{0,r}(z^M)$$

$$= z^{-(M-1)} \sum_{r=0}^{M-1} z^{-r} P_{0,r}(z^M)$$

Note z^M here! (owing to $\boxed{\uparrow M}$)

i.e. given the 0^{th} row of the $\mathbb{P}(z)$ matrix, the system's transfer function is

$$T(z) = z^{-(M-1)}[P_{0,0}(z^M) + z^{-1}P_{0,1}(z^M) + \dots + z^{-(M-1)}P_{0,M-1}(z^M)]$$

$$T(z) = z^{-(M-1)}[P_{0,0}(z^M) + z^{-1}P_{0,1}(z^M) + \cdots + z^{-(M-1)}P_{0,M-1}(z^M)]$$

Most General P.R. Conditions (necessary and sufficient)

Recall §1.7.3: sufficient condition for P.R. is $\mathbb{P}(z) = cz^{-m_0}\mathbb{I}$.

① To be free from aliasing, $\mathbb{P}(z)$ must be pseudo circulant.

② $T(z)$ must be a pure delay with possibly a constant multiplicative factor i.e. $T(z) = cz^{-d}$.

$\Longleftrightarrow$ iff the top row of $\mathbb{P}(z)$ is in the form of

$[0, 0 \ldots 0, cz^{-m_0}, 0, \cdots 0]$

Thus $\mathbb{P}(z) = cz^{-m_0}\begin{bmatrix} 0 & I_{M-r} \\ z^{-1}I_r & 0 \end{bmatrix}$ for some $r \in [0, M-1]$

When $r=0$, $\mathbb{P}(z) = \underline{I} \cdot cz^{-m_0}$ as in §1.7.3

The overall transfer function is

$$T(z) = c \cdot z^{-(M-1)} z^{-r} z^{-m_0 M}$$