On page 4 of Lecture 3(a) we posed the question of determining the range space of the map

$$u \mapsto L(u) = -\int_{t_0}^{t_1} \Phi(t_0,\sigma)B(\sigma)u(\sigma)\,d\sigma .$$

The following answers this.

Lemma: Let $\mathcal{U} = \{ u(\cdot) : [t_0, t_1] \to \mathbb{R}^m,\ u(\cdot) \text{ continuous} \}$. Define the inner product on $\mathcal{U}$

$$\langle u, v \rangle = \int_{t_0}^{t_1} u^T(t)\, v(t)\, dt .$$

Consider the map $L : \mathcal{U} \to \mathbb{R}^n$

$$L(u) = \int_{t_0}^{t_1} G(\sigma)\, u(\sigma)\, d\sigma$$

where $G(\cdot)$ is an $n \times m$ continuous matrix-valued function. Giving $\mathbb{R}^n$ the Euclidean inner product, the adjoint map $L^* : \mathbb{R}^n \to \mathcal{U}$ takes the form

$$\eta \mapsto L^*\eta , \quad (L^*\eta)(t) = G^T(t)\eta$$

and $\mathcal{R}(L) = \mathcal{R}(LL^*)$.

Proof: ($\Rightarrow$) Suppose $x_1 \in \mathcal{R}(LL^*)$.

Then $x_1 = LL^*\eta$ where,

$LL^*: \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$LL^* = \int_{t_0}^{t_1} G(\sigma) G^T(\sigma)\, d\sigma .$$

Let $u_1(\cdot)$ be given by

$$u_1(t) = G^T(t)\eta .$$

Then
$$L(u_1) = \int_{t_0}^{t_1} G(\sigma) G^T(\sigma)\eta\, d\sigma$$

$$= LL^*\eta$$

$$= x_1$$

Hence $x_1 \in \mathcal{R}(L)$.

We have shown $\mathcal{R}(LL^*) \subseteq \mathcal{R}(L)$.

($\Leftarrow$) We need to show that if $x_1 \notin \mathcal{R}(LL^*)$ then $x_1 \notin \mathcal{R}(L)$.

Given, $x_1 \notin \mathcal{R}(LL^*)$ $\exists$ $x_2 \in \operatorname{Ker}((LL^*)^*)$ $= \operatorname{Ker}(LL^*)$ (recall LL^* is symmetric), such that $x_1^T x_2 \neq 0$ [Fredholm alternative].

Suppose $x_1 \in \mathcal{R}(L)$ i.e. there exists $u_1(\cdot)$ such that

$$\int_{t_0}^{t_1} G(\sigma) u_1(\sigma)\, d\sigma = x_1 .$$

$$x_1^T x_2 = x_2^T x_1 = \int_{t_0}^{t_1} x_2^T G(\sigma) u_1(\sigma)\, d\sigma$$

$$\neq 0$$

But $$0 = x_2^T L^* x_2$$

$$= \int_{t_0}^{t_1} x_2^T G(\sigma) G^T(\sigma) x_2 \, d\sigma$$

$$= \int_{t_0}^{t_1} \| G^T(\sigma) x_2 \|^2 \, d\sigma$$

$$\Rightarrow \quad G^T(\sigma) x_2 = 0 \quad \forall\ \sigma \in [t_0, t_1] .$$

$$\Rightarrow \quad x_1^T x_2 = 0$$, a <u>contradiction</u>

Hence $x_1 \notin \mathcal{R}(L)$. ▨

Returning to the reachability question, we now conclude,

Theorem There exists a control $u(\cdot)$ that drives the system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

$$x(t_0) = x_0$$

to x_1 at time t_1 iff there is a vector $\eta \in \mathbb{R}^n$ such that

$$W(t_0, t_1)\,\eta = x_0 - \Phi(t_0, t_1)\,x_1 .$$

In that case, $u(\cdot)$ defined by

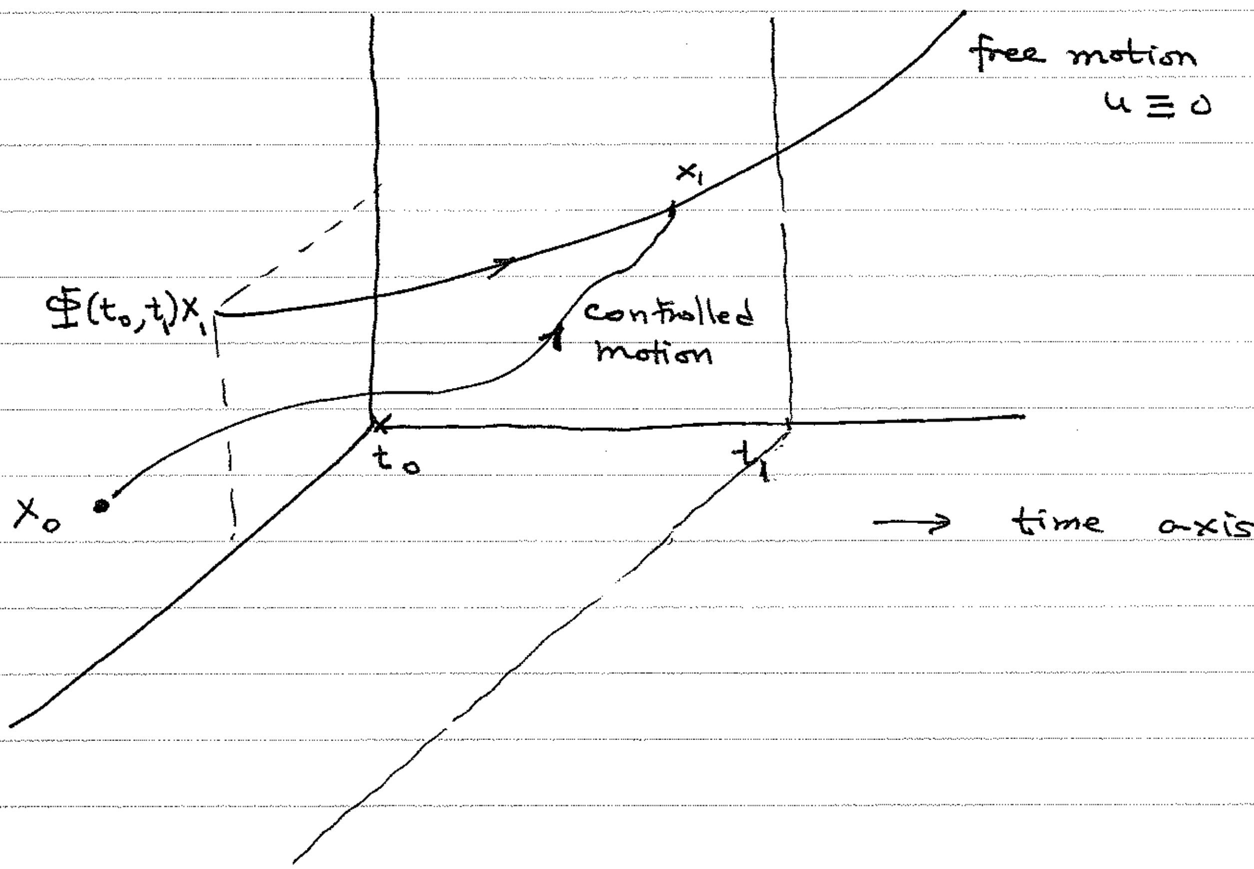
$$u(t) = -\left(\Phi(t_0, t)\,B(t)\right)^T \eta$$

$$= -B^T(t)\,\Phi^T(t_0, t)\,\eta$$

accomplishes the state transfer.

Corollary If $W(t_0, t_1)$ is invertible then any (x_0, t_0) can be transferred / driven to any (x_1, t_1), $t_1 > t_0$.

The following diagram (Brockett) illustrates the situation



The gap between x_0 and $\Phi(t_0, t_1)x_1$, and the uniqueness (given initial conditions) of solutions to the homogeneous equation indicates that free motion cannot transfer (x_0, t_0) to (x_1, t_1); only a suitably controlled motion can.

For the setting of time-invariant systems, i.e. $A(t) \equiv A$ a constant and $B(t) \equiv B$ a constant, the reachability Gramian $W(t_0, t_1)$ can be replaced by an alternative constant matrix with related properties.

Theorem Let $\dot{x} = Ax + Bu$ denote a linear time-invariant system.

Let

$$W(t_0, t_1) = \int_{t_0}^{t_1} e^{(t_0-\sigma)A} BB^T e^{(t_0-\sigma)A^T} d\sigma$$

$$= \int_{0}^{t_1-t_0} e^{-\sigma A} BB^T e^{-\sigma A^T} d\sigma$$

$$= W(0, t_1 - t_0)$$

and let

$$W_T = [B \; AB \; A^2B \; .. \; A^{n-1}B][B \; AB \; A^2B \; ... \; A^{n-1}B]^T$$

$$= \sum_{k=0}^{n-1} A^k BB^T A^{T^k}.$$

Then, for $t_1 > t_0$

$$\mathcal{R}(W(t_0, t_1)) = \mathcal{R}(W_T)$$

$$\mathrm{Ker}(W(t_0 t_1)) = \mathrm{Ker}(W_T)$$

Proof $(\Rightarrow)$ Suppose $x_1 \in \operatorname{Ker}(W(t_0,t_1))$.

Then

$$
\begin{aligned}
0 &= x_1^T W(t_0,t_1)\, x_1 \\
&= \int_{t_0}^{t_1} x_1^T e^{A(t_0-\sigma)} B B^T e^{A^T(t_0-\sigma)} x_1 \, d\sigma \\
&= \int_{t_0}^{t_1} \| B^T e^{A^T(t_0-\sigma)} x_1 \|^2 \, d\sigma
\end{aligned}
$$

$$\Rightarrow \quad B^T e^{A^T(t_0-\sigma)} x_1 = 0 \quad \forall \sigma \in [t_0, t_1].$$

differentiate $(n-1)$ times at $\sigma = t_0$ to obtain,

$$
\begin{aligned}
B^T x_1 &= 0 \\
B^T A^T x_1 &= 0 \\
&\vdots \\
B^T (A^T)^{n-1} x_1 &= 0
\end{aligned}
$$

$$\Rightarrow \quad \left(\sum_{k=0}^{n-1} A^k B B^T (A^T)^k \right) x_1 = 0$$

$$\Rightarrow \quad x_1 \in \operatorname{Ker}(W_T).$$

$(\Leftarrow)$ Suppose $x_1 \in \operatorname{Ker}(W_T)$

The Cayley Hamilton theorem says that

$$A^n + p_{n-1}A^{n-1} + p_{n-2}A^{n-2} + \cdots + p_0 I = 0$$

where $$\chi_A(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + p_{n-2}\lambda^{n-2} + \cdots + p_0$$

is the characteristic polynomial of A.
Using this repeatedly we conclude that

$$e^{tA} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$$

where $\alpha_i(\cdot)$ are functions dependent on the coefficients of the characteristic polynomial.

Then,

$$x_1^T W(t_0,t_1) = \int_{t_0}^{t_1} \sum_{k=0}^{n-1} \alpha_k(t_0-\sigma)\, x_1^T A^k B B^T e^{A^T(t_0-\sigma)}\, d\sigma$$

But $0 = x_1^T W_T$ (by hypothesis)

$$= x_1^T W_T x_1$$

$$= \sum_{k=0}^{n-1} \| B^T A^{T^k} x_1 \|^2$$

$\Rightarrow \quad x_1^T A^k B = 0 \qquad k = 0, 1, 2, \ldots, n-1$

$\Rightarrow \quad x_1^T W(t_0 t_1) = 0.$ (substituting in formula above)

$\Rightarrow \quad W(t_0,t_1)x_1 = 0$ (since $W(t_0 t_1) = W^T(t_0 t_1)$)

We have thus shown that

$$\operatorname{Ker}(W(t_0,t_1)) = \operatorname{Ker}(W_T)$$

By Fredholm alternative and the fact that $W_T = W_T^T$ and $W(t_0,t_1) = W^T(t_0,t_1)$, it follows that respective

Kernel and Range

are orthogonal to each other. Hence

$$\mathcal{R}(W(t_0,t_1)) = \mathcal{R}(W_T).$$

Corollary: $\mathcal{R}(W(t_0,t_1)) = \mathcal{R}([B \; AB \; \ldots \; A^{n-1}B])$

proof: W_T and $[B \; AB \; .. \; A^{n-1}B]$ have the same range. □

Definition We say that a linear constant coefficient system

$$\dot{x} = Ax + Bu$$

is Controllable if

$$\operatorname{rank} [B \; AB \ldots A^{n-1}B] = n.$$

Corollary: Controllability ⟹ any state can be driven to the origin in finite time.

Theorem Consider a controllable, single-input time-invariant linear system of the form

$$\dot{x} = Ax + bu$$

where $b \in \mathbb{R}^n$ A is $n \times n$. Then there is choice of basis in state space such that in this new basis the system can be brought to the canonical form

$$\dot{z} = A_c z + b_c u$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & & \cdots & p_{n-1} \end{bmatrix}$$

$$b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 1 \end{bmatrix}$$

where $\chi_A(s) = s^n + p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \cdots + p_1 s + p_0$.

Proof First note that a change of basis yields change of coordinates

$$z = Px,$$

where P is a nonsingular matrix and

$$\begin{aligned}\dot{z} &= P\dot{x} \\ &= P(Ax + bu) \\ &= PAP^{-1}z + Pbu \\ &= A_c z + b_c u\end{aligned}$$

Thus we seek P such that

$$A_c = PAP^{-1}$$

$$b_c = Pb.$$

By controllability

$$\operatorname{rank}[b, Ab, \ldots, A^{n-1}b] = n$$

i.e the vectors $\{b, Ab, \ldots, A^{n-1}b\}$ constitute a linearly independent set. Now consider the 'triangular' linear combinations of the form,

$$\begin{aligned} v_1 &= A^{n-1}b + p_{n-1}A^{n-2}b + \cdots + p_1 b \\ v_2 &= A^{n-2}b + p_{n-1}A^{n-3}b + \cdots + p_2 b \\ v_3 &= A^{n-3}b + p_{n-1}A^{n-4}b + \cdots + p_3 b \end{aligned}$$

$$\vdots$$

$$v_{n-1} = Ab + p_{n-1} b$$

$$v_n = b$$

The set of vectors $\{v_1, v_2, \dots, v_n\}$ is clearly linearly independent. (OBSERVE: Proceeding from v_n, a <u>new direction</u> $A^k b$ is added each time we move up the ladder through $v_{n-1}, v_{n-2}, \dots,$ to v_1.)

Let us express the matrix A and the matrix b in this new basis.

Clearly, since u takes values in $\mathbb{R}^1$ choosing '1' to bes basis for $\mathbb{R}^1$, we see

$$b.1 = \sum_{i=1}^{n} \tilde{b}_i v_i$$

$$= 0.v_1 + 0.v_2 + \dots + 0.v_{n-1} + 1.v_n$$

$\therefore$ Thus b has matrix representation

$$b_c = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

in the new basis.

By Cayley Hamilton theorem,

$$A^n = -p_{n-1}A^{n-1} - p_{n-2}A^{n-2} \cdots - p_0 I$$

$$\begin{aligned}
Av_1 &= A^n b + p_{n-1}A^{n-1}b + \cdots + p_1 Ab \cancel{+ p_0 b} \\
&= -p_0 b \\
&= -p_0 v_n
\end{aligned}$$

$$\begin{aligned}
Av_2 &= A^{n-1}b + p_{n-1}A^{n-2}b + p_{n-2}A^{n-3}b + \cdots + p_2 Ab \\
&= v_1 - p_1 b \\
&= v_1 - p_1 v_n
\end{aligned}$$

$$\begin{aligned}
Av_3 &= A^{n-2}b + p_{n-1}A^{n-3}b + \cdots + p_3 Ab \\
&= v_2 - p_2 b \\
&= v_2 - p_2 v_n
\end{aligned}$$

$\vdots$

$$\begin{aligned}
Av_{n-1} &= A^2 b + p_{n-1}Ab \\
&= v_{n-2} - p_{n-2} b \\
&= v_{n-2} - p_{n-2} v_n
\end{aligned}$$

$$\begin{aligned}
Av_n &= Ab \\
&= v_{n-1} - p_{n-1} b \\
&= v_{n-1} - p_{n-1} v_n
\end{aligned}$$

It follows that the matrix representation of the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ in the <u>new basis</u> $\{v_1, v_2, \ldots, v_n\}$ is

given by,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & \cdots & -p_{n-1} \end{bmatrix}.$$

This completes ~~the proof~~.

(We did not write down P explicitly. Instead we invented a basis and <u>expressed</u> A as A_c in this new basis, and similarly b as b_c in this new basis. With $\{e_1, e_2, \ldots, e_n\}$ as the standard basis in $\mathbb{R}^n$, the new basis $\{v_1, v_2, \ldots, v_n\}$ can be expressed as the matrix

$$\begin{bmatrix} v_1, v_2, \ldots, v_n \end{bmatrix} = \begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 & \cdots & p_{n-1} & 1 \\ p_2 & p_3 & p_4 & \cdots & 1 & \\ \vdots & & & & & \\ & & p_{n-1} & 1 & & 0 \\ p_{n-1} & 1 & & & & \\ 1 & & & & & \end{bmatrix}$$

$$= T$$

the change of basis from $\{e_1, \ldots e_n\}$ to $\{v_1, v_2, \ldots, v_n\}$

Adopting the convention introduced in Lecture 1 (a), page 10,

$$P = T^{-1}$$

Verify these steps.)

Remark The pair $[A_c, b_c]$ or equivalently the system

$$\dot{z} = A_c z + b_c u$$

Constitute the control canonical form. Observe that, if we let

$$y(t) = z_1(t),$$

then

$$\frac{d^n}{dt^n} y(t) + p_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \dots + p_1 \frac{dy}{dt} + p_0 y$$

$$= u(t)$$

Thus the control canonical form reveals structure of an n^{th} order differential equation underlying the original system.