An orthogonality result, and optimality

In Lecture 3(b) we saw that the problem of tranferring a state x_0 at time t_0, to a state x_1 at time $t_1 > t_0$, is equivalent to solving the equation for $u(\cdot)$

$$x_0 - \Phi(t_0,t_1)x_1 = L(u) = -\int_{t_0}^{t_1} \Phi(t_0,\sigma)B(\sigma)u(\sigma)d\sigma.$$

A solution exists iff $\exists\ \eta \in \mathbb{R}^n$ such that

$$x_0 - \Phi(t_0,t_1)x_1 = LL^*\eta$$

$$= W(t_0,t_1)\eta$$

where $$W(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\sigma)B(\sigma)B^T(\sigma)\Phi^T(t_0,\sigma)d\sigma$$

and $$(L^*\eta)(t) = -B^T(t)\Phi^T(t_0,t)\eta$$

If such an η exists then,

$$u_0(t) = \cancel{\#}\, L^* \eta\,(t)$$

accomplishes the transfer.

If $u_1(\cdot)$ is any other continuous control that also accomplishes the state transfer, then the following orthogonality result holds

$$\langle u_1 - u_0, u_0 \rangle_{\mathcal{U}} = \int_{t_0}^{t_1} \left(u_1(\sigma) - u_0(\sigma)\right)^T u_0(\sigma)\, d\sigma$$

$$= 0.$$

<u>Proof</u> Since $u_1(\cdot)$ accomplishes transfer

$$x_0 - \Phi(t_0, t_1) x_1 = L(u_1) = W(t_0, t_1)\eta$$

$$\langle u_1 - u_0, u_0 \rangle_{\mathcal{U}} = \langle u_1, u_0 \rangle_{\mathcal{U}} - \langle u_0, u_0 \rangle_{\mathcal{U}}$$

$$= \langle u_1, L^*\eta \rangle_{\mathcal{U}} - \langle L^*\eta, L^*\eta \rangle_{\mathcal{U}}$$

$$= \langle Lu_1, \eta \rangle_{\mathbb{R}^n} - \langle LL^*\eta, \eta \rangle_{\mathbb{R}^n}$$

$$= \langle W\eta, \eta \rangle_{\mathbb{R}^n} - \langle W\eta, \eta \rangle_{\mathbb{R}^n}$$

$$= 0 \qquad \square$$

Corollary u_0 solves the <u>constrained optimization</u> problem

$$\text{Min } \tfrac{1}{2}\langle u, u\rangle_{\mathcal{U}}$$
$$L(u) = x_0 - \Phi(t_0, t_1)x_1$$

Proof By symmetry of inner product, and the orthogonality property -

$$\langle u_0, u_1 - u_0\rangle_{\mathcal{U}} = 0$$

Now, by positive definiteness of inner product,

$$0 \leq \langle u_1 - u_0, u_1 - u_0\rangle_{\mathcal{U}}$$

$$= \langle u_1 - u_0, u_1\rangle_{\mathcal{U}} - \langle u_1 - u_0, u_0\rangle_{\mathcal{U}}$$

$$= \langle u_1 - u_0, u_1\rangle_{\mathcal{U}} - 0 \qquad \text{(orthogonality)}$$

$$= \langle u_1, u_1\rangle_{\mathcal{U}} - \langle u_0, u_1\rangle_{\mathcal{U}}$$

$$= \langle u_1, u_1\rangle_{\mathcal{U}} - \langle u_0, u_0 + u_1 - u_0\rangle_{\mathcal{U}}$$

$$= \langle u_1, u_1\rangle_{\mathcal{U}} - \langle u_0, u_0\rangle_{\mathcal{U}} - \langle u_0, u_1 - u_0\rangle_{\mathcal{U}}$$
$$= \langle u_1, u_1\rangle_{\mathcal{U}} - \langle u_0, u_0\rangle_{\mathcal{U}} \qquad \text{(orthogonality)}$$

Thus $\langle u_1, u_1 \rangle_{\mathcal{U}} \geq \langle u_0, u_0 \rangle_{\mathcal{U}}$ for all

u_1 s.t. $L(u_1) = x_0 - \Phi(t_0, t_1) x_1$.

Hence u_0 is ~~the~~ a global minimizer of the constrained optimization problem $\square$

<u>Remark</u> u_0 depends on η with

$$W(t_0, t_1)\eta = x_0 - \Phi(t_0, t_1) x_1$$

η is not unique unless $W(t_0, t_1)$ is invertible. Hence u_0 is not unique unless $W(t_0, t_1)$ is invertible. But ~~the~~ all such u_0 have the same <u>minimum cost</u>,

$$\langle u_0, u_0 \rangle_{\mathcal{U}} = \langle L^* \eta, L^* \eta \rangle_{\mathcal{U}}$$

$$= \langle L L^* \eta, \eta \rangle_{\mathbb{R}^n}$$

$$= \eta^T W(t_0, t_1) \eta$$

<u>Remark</u> If $W(t_0, t_1)$ is invertible, then

optimal cost $= (x_0 - \Phi(t_0, t_1) x_1)^T W(t_0, t_1)^{-1} (x_0 - \Phi(t_0, t_1) x_1)$

It is pleasing to note that the Gramian approach leads to an optimal control. In the case when $W(t_0, t_1)$ is invertible, the minimum cost is given by the quadratic form

$$(x_0 - \Phi(t_0, t_1)x_1)^T \, W(t_0, t_1)^{-1} \, (x_0 - \Phi(t_0, t_1)x_1).$$

Recall (Home Work Set 6, Problem 4) $W(t, t_1)^{-1}$ satisfies a Riccati differential equation

$$\frac{dP}{dt} = -A^T P - PA + PBB^T P.$$

Such equations (and variants) arise in the full theory of <u>linear optimal control with quadratic cost functionals</u>.