Classification of controllable pairs $[A, B]$.

Let $n \geq 1$, $m \geq 1$ be fixed. Let $\Sigma_{n,m}$ denote all possible pairs of matrices $[A, B]$. Clearly this is a vector space of dimension $n^2 + nm$. A pair $[A, B]$ is ~~—~~ uncontrollable iff

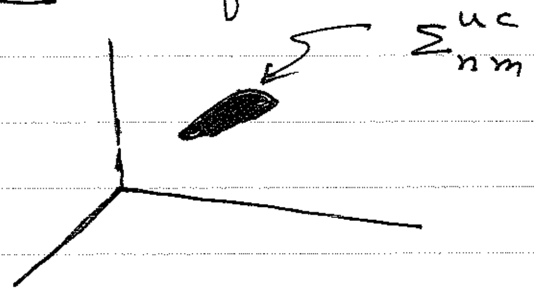
$$\det\left(\sum_{k=0}^{n-1} A^k B B^T A^{T\,k}\right) = 0 .$$

A pair $[A, B]$ is controllable iff

$$\det\left(\sum_{k=0}^{n-1} A^k B B^T A^{T\,k}\right) > 0 .$$

Thus $\Sigma_{n,m}$ breaks up into two disjoint pieces $\Sigma^{uc}_{n,m}$ the uncontrollable systems and $\Sigma^{c}_{n,m}$ of the controllable systems.

The piece $\Sigma^{uc}_{n,m}$ is a "thin set" since it satisfies an algebraic condition of a vanishing determinant.



This picture illustrates that what is outside the dark, thin blob $\Sigma^{uc}_{n,m}$, makes up the controllable systems.

We are interested in showing that $\Sigma^{c}_{n,m}$ falls into a finite ~~set~~ number of disjoint regions (also called equivalence classes, orbits etc) such that if $[A, B]$ and $[\tilde{A}, \tilde{B}]$ belong to the same region, then there is a combination of transformation of the form

(i) $[A, B] \longmapsto [PAP^{-1}, PB]$ — Change of basis in state space

P $n \times n$ invertible

(ii) $[A, B] \longmapsto [A + BK, B]$ — state feedback

K $m \times n$

(iii) $[A, B] \longmapsto [A, BQ]$ — change of basis in input space

Q $m \times m$ invertible

that take $[A, B]$ into $[\tilde{A}, \tilde{B}]$. They constitute the <u>feedback group</u>.

An <u>example</u> of such a <u>combination</u> is

$$[A, B] \mapsto [PAP^{-1} + PBK,\ PBQ].$$

Each of the transformations can be <u>undone</u> (relevant inverses exist). Hence, $[A, B]$ and $[\tilde{A}, \tilde{B}]$ are <u>equivalent</u> if they each can be transformed into a <u>standard</u> or <u>canonical form</u>.

For instance, when $m = 1$, any two controllable pairs in $\Sigma^c_{n,1}$, $[A, b]$ and $[\tilde{A}, \tilde{b}]$, can be transformed under the feedback group into the canonical form

$$A_c = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 0 & 0 & 0 & & 0 \end{bmatrix} \qquad b_c = \begin{bmatrix} 0 \\ 0 \\ \\ 1 \end{bmatrix}$$

We saw this earlier under the setting

of single-input system eigenvalue placement theorem.

Note: We did not make use of change basis in input space for $m=1$ since in that case it is subsumed by the special case of change of basis in state space with

$$P = q I \qquad q \neq 0$$

Since,

$$\left[(qI) A (qI)^{-1}, \ (qI) b \right]$$

$$= [A, \ bq]$$

In the multi-input setting the change of basis in input space really makes a difference.

Finding invariants under feedback group

Let $L_j = \operatorname{im} [B \ AB \ \cdots \ A^j B]$

$\quad = $ linear subspace of $\mathbb{R}^n$ spanned by columns of $[B \ AB \cdots A^j B]$.

By hypothesis $L_{n-1} = \mathbb{R}^n$ (controllability)

Clearly

$$\mathrm{im}(B) = L_0 \subseteq L_1 \subseteq L_2 \cdots \subseteq L_{n-1} = \mathbb{R}^n$$

Let $\mathbb{R}^n$ be given an inner product, say Euclidean inner product.

Let Λ_j = orthogonal complement of L_{j-1} in L_j

$$= \{ v \in L_j : \langle v, w \rangle = 0 \ \ \forall\, w \in L_{j-1} \}$$

Here $\langle v, w \rangle$ = inner product of v and w.

Then

$$L_j = L_{j-1} \oplus \Lambda_j$$

direct sum of orthogonal vectors

$$\mathbb{R}^n = L_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \cdots \oplus \Lambda_{n-1}$$

Let $r_0 = \dim L_0 = \mathrm{rank}(B)$

$r_j = \dim(L_j) - \dim(L_{j-1})$

$= \dim(\Lambda_j)$ $\quad j = 1, 2, \ldots, n-1$

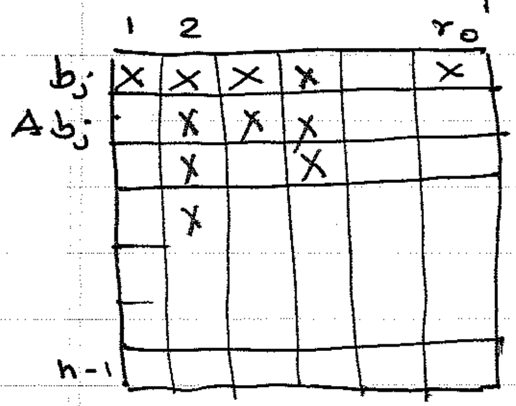
Clearly $r_0 + r_1 + r_2 \cdots + r_{n-1} = n$.

Clearly $r_0 > 0$ and $r_0 \le m$.

Clearly $r_j \le m$ for $j \le n-1$

By re-ordering the columns of B, equivalently, right multiplying B by a suitable nonsingular Q, we can be sure that columns 1 through r_0 of B are linearly independent. They constitute a basis for L_0.

Create the following Young diagram with r_0 columns and n rows.



Fill first row with crosses, r_0 in number to mark selection of columns 1 through r_0 of B as basis of L_0.

Mark a cross in second row in j^{th} place only if Ab_j is linearly independent of all vectors $b_1, b_2, \ldots, b_{r_0}, Ab_1, \ldots, Ab_{j-1}$.

Total # crosses in second row $= r_1 = \dim(\Lambda_1) = \dim L_1 - \dim L_0$.

In i^{th} row mark a cross in j^{th} place only if $A^i b_j$ is linearly independent of all vectors $b_1, b_2, \ldots, b_{r_0}, Ab_1, \ldots, Ab_{r_0}, \ldots, A^i b_1, \ldots, A^i b_{j-1}$.

It follows that

$$r_0 \geq r_1 \geq r_2 \cdots \geq r_{n-1}$$

We have picked basis S (by marking crosses) of vectors $A^j b_i$ with the property that if $A^j b_i \notin S$ then $A^{j+\ell} b_i \notin S$ for $\ell > 0$.

We now associate with every b_i a number κ_i such that $A^j b_i \in S$ for $0 \leq j \leq \kappa_i - 1$ but $A^{\kappa_i} b_i \notin S$. Again by re-ordering the $\overbrace{\text{}}^{\text{first } r_0}$ columns of B, we can achieve

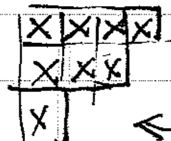
$$\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_{r_0} .$$

But κ_i's are simply column sums of the number of crosses. Hence

$$\sum_{i=1}^{r_0} \kappa_i = \sum_{j=0}^{n-1} r_j = n .$$

Sequence $\kappa_1, \kappa_2, \cdots, \kappa_{r_0}$ defines a partition of the integer n.

Remark (i) κ_i's determine r_j's and vice-versa



(ii) After reordering in B,

$\{ A^j b_i \; : \; 1 \leq i \leq r_0 , \; 0 \leq j \leq \kappa_i - 1 \}$

Young diagram

is a basis for $\mathbb{R}^n$.

(iii) under $[A, B] \to [PAP^{-1}, PB]$

$$L_j \to P L_j \quad \text{so}$$

list $r_0, \ldots, r_{n-1}$ is invariant, and hence $\kappa_1, \ldots, \kappa_{r_0}$ ordered list of κ's is also invariant.

(iv) under $[A, B] \to [A, BQ]$

$L_i \to L_i$ so remarks as in (iii) apply regarding invariance.

(v) $[A, B] \longrightarrow [A+BK, B]$

$$(A+BK)^j \cancel{} = A^j \cancel{} B + G$$

where G is a matrix whose columns are contained in L_{j-1}.

Hence L_j invariant under $A \mapsto A+BK$

.... r_j, κ_i invariant.

Canonical Form

1. Write down pyramidal basis

$$S = \{ b_i, \ldots, A^{\kappa_i - 1} b_i, \ldots \mid i \in M' \} \quad \text{for } \mathbb{R}^n.$$

$M' \subseteq \{1, 2, \ldots, m\}$ containing exactly $r_0 = \text{rank}(B)$ elements

Linear dependence relations:

$$A^{\kappa_i} b_i = - \sum_{j \in M'} \sum_{k=1}^{\kappa_j - 1} \alpha_{ijk} A^k b_j$$

$$\forall \, i \in M'$$

determine unique coefficients α_{ijk}

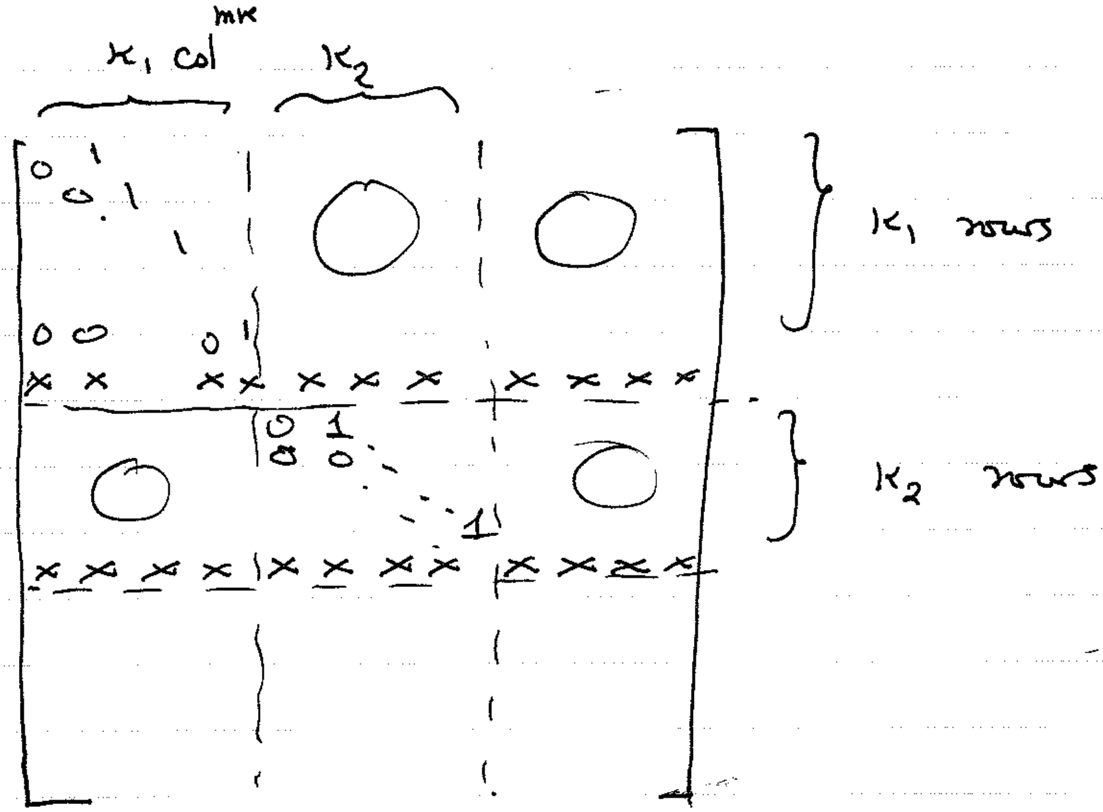
2. New basis

$$\begin{aligned} e_{j1} &= b_j \\ e_{j2} &= A b_j + \alpha_{jj1} b_j \\ &\vdots \\ e_{j\kappa_j} &= A^{\kappa_j - 1} b_j + \alpha_{jj1} A^{\kappa_j - 2} b_j + \cdots + \alpha_{jj\kappa_j - 1} b_j \end{aligned}$$

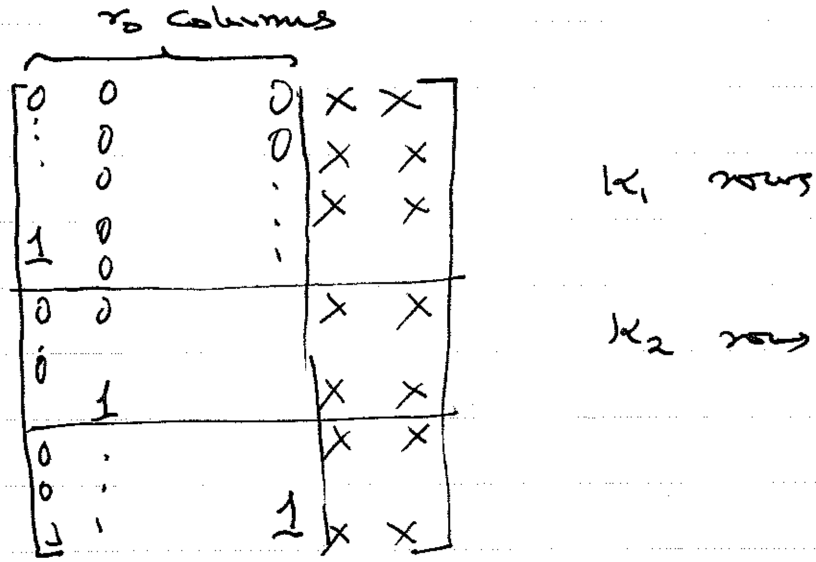
$$j \in M'$$

Direct calculation shows that in this <u>new basis</u> A and B take canonical forms

$A =$



$B =$



The elements marked $\times$ in A are given by α_{ijk}. The last $(m-r_0)$ columns of B are linearly dependent on the first r_0 columns.

By applying another $B \to BQ$ we can make last $(m-r_0)$ columns equal to 0.

By <u>now</u> applying feedback $A \mapsto A+BK$ with appropriate rows of K containing $-\alpha_{ijk}$ we can <u>kill</u> off all crosses in A. We have,

<u>Theorem</u> [Brunovsky]

$[A, B] \in \Sigma^c_{n,m}$. Then there exist positive integers $\kappa_1, \kappa_2, \cdots, \kappa_{r_0}$ uniquely determined by $[A, B]$ such that $[A, B]$ is equivalent under the feedback group to $[A_0, B_0]$ where

$$A_0 = \operatorname{diag}(E_{\kappa_1}, E_{\kappa_2}, \cdots, E_{\kappa_{r_0}})$$

$$B_0 = \begin{array}{c} \left[e_{\kappa_1} \;\; e_{\kappa_2+\kappa_1} \cdots e_{\kappa_j+\kappa_{j-1}+\cdots+\kappa_1}, \cdots, e_n \;\middle|\; 0 \right] \\ \uparrow \qquad\quad \uparrow \qquad\qquad\qquad\qquad\qquad\quad \uparrow \\ \text{column } 1 \quad 2 \qquad\qquad\qquad\qquad \text{column } r_0 \end{array}$$

$$E_\kappa = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\kappa \times \kappa}$$

Since there are finitely partitions, there are finitely many classes ■