ENEE 661 Nonlinear Control Systems

Lecture 5 (part ii)

PSK
03.10.00

For a nonlinear system $\dot{x} = f(x)$, with equilibrium x_e, we define the *region of attraction* of x_e denoted $R_A(x_e)$ to be the set

$$R_A(x_e) = \{ x_0 : \phi(t, x_0) = \text{sol}^n \text{ to } \dot{x}=f(x) \to x_e \text{ as } t \to \infty \}$$

If x_e is an asymptotically stability equilibrium then one can prove that $R_A(x_e)$ is an open, invariant set containing x_e. When is it the whole state space? Clearly, if $R_A(x_e) = \mathbb{R}^n$, then x_e is the unique equilibrium. Additionally, one can rule out all other invariant sets (e.g. periodic orbits). The following is a set of sufficient conditions.

See pages 10 & 11 of these notes

Theorem (Barabashin - Krasovskii)

Let $x = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that

(i) $V(0) = 0$, $V(x) > 0$ $x \neq 0$.

(ii) $V(x) \to \infty$ as $\|x\| \to \infty$ (radial unboundedness)

(iii) $\dot{V}(x) < 0$ $\forall x \neq 0$

Then $x = 0$ is globally asymptotically stable ($\Rightarrow R_A(0) = \mathbb{R}^n$)

No loss of generality in picking $x_e = 0$; recall change of coordinates trick.

Proof: Let $p \in \mathbb{R}^n$, $V(p) = c > 0$.
From (ii) there exists $r > 0$ s.t.

PSK
03.10.00

$$\{x \in \mathbb{R}^n \mid V(x) \le c\} = \Omega_c \subset B_r.$$

Thus Ω_c is bounded, (and closed). By LaSalle's invariance principle $x(t)$, a trajectory beginning in Ω_c, $\to$ M = largest invariant set $\subset E = \{x \mid \dot V = 0\}$, as $t \to \infty$.

By (iii) $M = \{0\}$.

Thus 0 is asymptotically ~~stability~~ stable and globally attractive. $\square$

Chetaev = Cetaev = Chetayev

The idea of using "energy-like" function D can be used also to prove instability theorems. A fundamental result is due to Nikolai Gurryewich Chetaev (1902–1959), a Soviet mechanician, who held the chair of Theoretical Mechanics at Moscow State University.

Theorem (Cetaev)

Let $x = 0$ be an equilibrium point of $\dot x = f(x)$. Let $V : D \to \mathbb{R}$ be a C^1 function defined on an open, connected subset D of $\mathbb{R}^n$ which contains 0, and such that:

(i) $V(0) = 0$ and for every $\varepsilon > 0$, there exists $x_0 \in B_\varepsilon(0)$ such that $V(x_0) > 0$.

(ii) Let $U = \{x \in B_r(0) \mid V(x) > 0\}$ for $B_r \subset D$ and

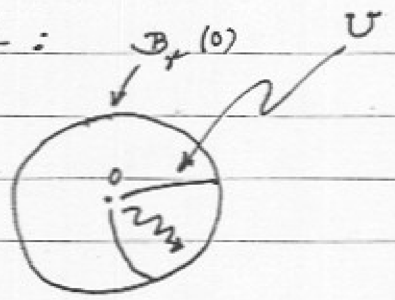
PSK
03.10.00

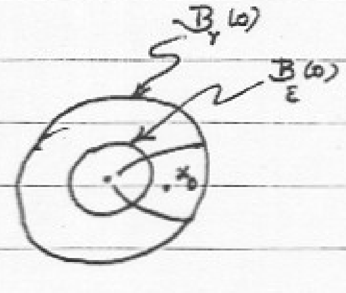
$\phi : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing with $\phi(0) = 0$.

Then $x = 0$ is an unstable equilibrium

(remark: such a function ϕ is called a class $\mathcal{K}$ function)

Proof:





Let $x_0 \in \text{int}(U)$ and $V(x_0) = a > 0$. Suppose $\|x_0\| = r_0$. Consider the annulus

$$\{x : \epsilon \le \|x\| \le r\}$$

where $\varepsilon > 0$ is such that $V(x) \le \frac{a}{2}$ for $\|x\| \le \varepsilon$. Such an ε exists by continuity of V at the origin. Then, since $\dot{V}(x(t)) > 0$ and $V(x_0) = a$, it follows that a trajectory must not enter B_ε.

The annulus $\{x : \varepsilon \le \|x\| \le r\}$ is a closed and bounded set and $\dot{V}(x) > 0$ in U. Since $\dot{V}(x) \geq \phi(\|x\|)$ a class $\mathcal{K}$ function $\dot{V} \geq \gamma > 0$ for some γ. We can take

$$\gamma = \inf \{ \phi(\lambda) / \lambda \in [\varepsilon, r] \}.$$

Then,

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))\, ds$$

$$\geq a + \int_0^t \gamma \cdot ds$$

$$= a + \gamma t.$$

But $V(\cdot)$ is bounded in U. Thus $x(t)$ cannot remain forever in U. In fact it leaves U at ~~the~~ latest by $T = \frac{1}{\gamma}\left(\sup_U V - a\right)$.

Now $x(t)$ cannot leave U through the surface $\{x : V(x) = 0\}$ since $V(x(t)) \geq a > 0$. Hence it must leave through the sphere $\|x\| = r$. Since this can happen for arbitrarily small $\|x_0\|$, the origin is unstable. $\boxtimes$

<We call such a V a <u>Cetaev function</u>>

<u>Example</u>

Consider the planar system

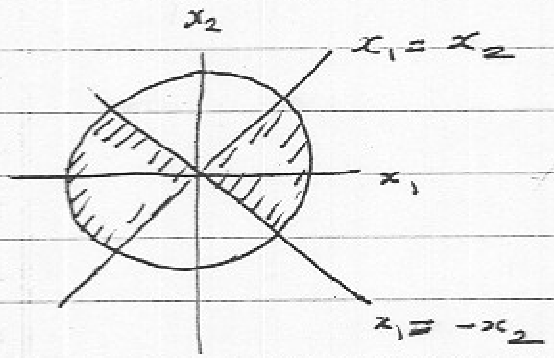
$$\dot{x}_1 = x_1 + g_1(x)$$
$$\dot{x}_2 = -x_2 + g_2(x)$$

where $|g_i(x)| \leq k \|x\|_2^2$ in a nbhd D of 0, ($\Rightarrow g_i(0) = 0$ and thus 0 is an equilibrium point). Let $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$.

PSK
03.10.08

On the line $x_2 = 0$, i.e. the x_1 axis, $V(x) > 0$ at points arbitrarily close to the origin.

Pick
$$U = \{ (x_1, x_2) \in B_r(0) \mid V(x_1, x_2) > 0 \}$$
i.e. the shaded area in the adjoining figure.



$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x).$$

But
$$|x_1 g_1(x) - x_2 g_2(x)| \le \sum_{i=1}^{2} |x_i| \, g_i(x)$$
$$\le 2k \, \|x\|_2^3 \quad \text{(by hypothesis)}$$

Hence
$$\dot{V}(x) \ge x_1^2 + x_2^2 - |x_1 g_1(x) - x_2 g_2(x)|$$
$$\ge x_1^2 + x_2^2 - 2k \, \|x\|_2^3$$
$$= \|x\|_2^2 \, (1 - 2k \, \|x\|_2).$$

Choose $r > 0$ such that $B_r(0) \subset D$ and $r < \frac{1}{2k}$. Then $\dot{V}(x) > 0$ on U.

By Cetaev's theorem the origin is unstable. (Question: what is ϕ here?) $\boxtimes$

PSK
03.10.00

Example

$$M\ddot{x} + (S + \varepsilon R)\dot{x} + Kx = 0$$

$M = M^T > 0$, $S = -S^T$, $R = R^T > 0$, $K = K^T$ indefinite.

Let $p = M\dot{x}$

$$\dot{x} = M^{-1}p$$

$$\dot{p} = -(S + \varepsilon R)M^{-1}p - Kx$$

$\varepsilon > 0$.

$$H = \frac{1}{2}p^T M^{-1} p + \frac{1}{2}x^T K x$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p}$$

$$= Kx \cdot M^{-1}p + M^{-1}p \cdot ((-S - \varepsilon R)M^{-1}p - Kx)$$

$$= -\varepsilon\, p \cdot M^{-1}RM^{-1}p \qquad (\because S = -S^T)$$

Pick the Cetaev function to ~~be~~ be $V = -H$. Convince yourself, by argument, as in the previous example (pick U properly), that $(0,0)$ is unstable in $\mathbb{R}^n \times \mathbb{R}^n$.

PSK
03.14.00

There are instability results due to Lyapunov that precede the work of Cetaev and were motivating influences on Cetaev's work.

<u>Theorem</u> (Lyapunov - instability **I**)

If V is C' in a domain $D \subset \mathbb{R}^n$ with $0 \in D$, $f(0) = 0$ and $V(0) = 0$. Also $\dot{V}(x) > 0$ $x \neq 0$ in D, and $V(x)$ assumes positive values arbitrarily near 0, then 0 is an unstable equilibrium.

<u>Proof</u>: Without loss of generality, let D be a bounded domain. V is bounded in D. Let $r \leq R$, $B_r(0) \subset D$. Then $\exists\ x_0 \in B_r(0)$ such that $V(x_0) > 0$.

Since $\dot{V} > 0$, $V(x(t))$ can only increase and so $x(t) \not\to 0$. In fact, since $\dot{V} \geq m > 0$, $x(t) \not\to$ fixed point in $B_R(0)$. $V(x(t))$ must increase indefinitely and hence $x(t)$ must eventually leave $B_R(0)$ via some point in the boundary $\partial B_R(0)$. $\boxtimes$

<u>Theorem</u> (Lyapunov - Instability II)

Same hypotheses on V as in (I) above, and $\dot{V} = \lambda V + V^*$ where $V^* \geq 0$ in D and $\lambda > 0$. Then 0 is an unstable equilibrium.

PSK
03.14.00

Proof: Let $x_0 \in B_r(0)$ as in previous. $V(x_0) > 0$.
$x(t) \triangleq \phi(t, x_0)$.

$$\dot{V} = \lambda V + V^* \Rightarrow$$

$$\dot{\overbrace{e^{-\lambda t} V}} = e^{-\lambda t} V^* \geq 0.$$

Hence along $x(t)$, $V \geq e^{\lambda t} V(x_0)$.
V increases indefinitely along $x(t)$
$\Rightarrow$ instability as in previous theorem ☒

<u>Some special Lyapunov functions</u>

Let $V = V_p(x) + V_{p+1}(x) + \cdots$
be defined in a nbhd of 0, where V_k is a homogeneous polynomial of degree k. Then the sign of V in a suitable nbhd Ω of the origin is the same as the sign of V_p.

<u>Lemma</u>: If p is odd V cannot be a Lyapunov function.

<u>Proof</u>: $x_1 = x_n u_1$; $x_2 = x_n u_2$; $\cdots$; $x_{n-1} = x_n u_{n-1}$
$\Rightarrow V_p = x_n^p V_p(u_1, u_2, \cdots, u_{n-1}, 1)$. Keeping the u_i fixed, the sign of V_p will be the sign of x_n^p or $-x_n^p$ (one of the two but not both). Since p is odd x_n^p or $-x_n^p$ may assume both positive and negative values near 0. So

V is not positive definite. We are tacitly assuming that we have chosen u_i s.t. $V_p(u_1, u_2, \ldots, u_{n-1}, 1) \neq 0$. This is always possible since $V_p \neq 0$. $\boxtimes$

Note: Such power series expansions as in V above may not be defined, but we might still have a positive definite C' Lyapunov function e.g.

$$V(x) = \begin{cases} x^2 & x \geq 0 \\ x^4 & x \leq 0 \end{cases}.$$

Lemma (Invariant Sets)

If $x = 0$ is an asymptotically stable equilibrium point, then its region of attraction $R_A(0)$ is an open, invariant set. Moreover $\partial R_A(0)$ the boundary of $R_A(0)$, is formed by trajectories.

Proof: Let $\phi(s, x)$ denote the solution to $\dot{y} = f(y)$, with initial condition $y(0) = x$, $s \in \mathbb{R}$. We wish to show that $\phi(s, x) \in R_A(0)$ whenever $x \in R_A(0)$, $\forall\, s \in \mathbb{R}$.

PSK
03.14.00

By the semigroup property of solutions to o.d.e's

$$\phi(t, x(s)) = \phi(t, \phi(s, x))$$
$$= \phi(t+s, x)$$

$$\lim_{t\to\infty} \phi(t, x(0)) = \lim_{t\to\infty} \phi(t+s, x)$$
$$\in R_A(0) \quad \text{by definition.}$$

Thus $\phi(s, x) \in R_A(0)$ $\forall$ $s \in \mathbb{R}$, whenever $x \in R_A(0)$. (INVARIANCE)

To prove openness of $R_A(0)$, let $p \in R_A(0)$. Let $T > 0$ be sufficiently large that

$$\|\phi(T, p)\| < \frac{a}{2}$$

where $a > 0$ is such that $\{x : \|x\| < a\} \subset R_A(0)$

We can choose b small enough such that $\forall$ $q \in \{x : \|x - p\| < b\}$, the solution $\|\phi(T, p) - \phi(T, q)\| < \frac{a}{2}$.

Hence $\|\phi(T, q)\| \leq \|\phi(T,p) - \phi(T,q)\| + \|\phi(T,p)\|$
$< a.$

$\Rightarrow$ $\phi(T, q) \in R_A(0)$
$\Rightarrow$ $\lim_{t\to\infty} \phi(t, q) = 0 \Rightarrow q \in R_A(0).$

11

PSK
03.14.00

For any open, invariant set M

$x \in \partial M \Rightarrow \{x_n\} \subset M$ s.t. $\lim_{n\to\infty} x_n = x$.

Hence $\{\phi(t, x_n) : t \in \mathbb{R}\} \subset M$.

~~Base~~ $\lim_{t\to\infty} \phi(t, x_n) = \phi(t, x)$

$\Rightarrow$ $\phi(t, x)$ is an accumulation point of M; $\forall\, t \in \mathbb{R}$.

But $\phi(t,x) \notin M$, since $x \in \partial M$ & M is open. Therefore $\phi(t,x) \in \partial M$ $\forall\, t \in \mathbb{R}$. Thus ∂M is made up of trajectories.