## Lemma

Let $\dot{x}(t) = f(t, x)$ and $f(t, 0) \equiv 0$. Suppose $f$ is piecewise continuous in $t$ and satisfies the Lipschitz condition in $x$

$$\| f(t, x_1) - f(t, x_2) \|_2 \le L \, \| x_1 - x_2 \|_2 . \qquad \forall \, x_i \in B_r(0)$$

Then

$$\| x(t_0) \|_2 \, e^{-L(t - t_0)} \le \| x(t) \|_2 \le \| x(t_0) \|_2 \, e^{L \cdot (t - t_0)}$$

$\square$

**Proof:** From $\langle x, x \rangle = \| x \|_2^2$, it follows that

$$\frac{d}{dt} \| x \|_2^2 = 2 \, \| x \|_2 \, \frac{d}{dt} \| x \|_2$$

$$= 2 \langle \dot{x}, x \rangle$$

Hence $\| x \|_2 \left| \frac{d}{dt} \| x \|_2 \right| = | \langle \dot{x}, x \rangle |$

$$\le \| x \|_2 \, \| \dot{x} \|_2 \qquad \text{(Cauchy-Schwartz)}$$

Thus, $\left| \frac{d}{dt} \| x \|_2 \right| \le \left\| \frac{d x}{dt} \right\|_2 \le \| f(t, x) \|_2$

$$\le L \, \| x \|_2$$

$$\forall \, x \in B_r(0)$$

Then $-L \, \| x \|_2 \le \frac{d}{dt} \| x \|_2 \le L \, \| x \|_2$

By Bellman - Gronwall inequality

$$\| x(t_0) \|_2 \, e^{-L(t - t_0)} \le \| x(t) \|_2 \le \| x(t_0) \|_2 \, e^{L(t - t_0)}$$

as long as $x \in B_r(0)$

(OVER)

$$\Rightarrow \quad \tilde{r}(t) \leq \tilde{y}(0) - \tilde{y}(0)\, e^{-tL}$$

$$\tilde{y}(t) \leq \tilde{y}(0) - \tilde{r}(t)$$

$$\leq \tilde{y}(0) - \tilde{y}(0) + \tilde{y}(0) e^{-tL}$$

$$= \tilde{y}(0) e^{-tL}$$

$$-y(t) \leq -y(0)\, e^{-tL}$$

$$\Rightarrow \quad y(t) \geq y(0) e^{-tL}$$

## Exponential Stability Theorem and its Converse.

Consider $\dot{x} = f(t,x)$ with ~~f~~ $f$ $C^1$ in $x$ and piecewise continuous in $t$. Then the two statements below are equivalent:

(i) $x=0$ locally exponentially stable equilibrium point of $\dot{x} = f(t,x)$ ; i.e if $x \in B_h(0)$ for $h$ small enough, there exist $m, \alpha > 0$ such that

$$\| \phi(\tau, t, x) \|_2 \leq m e^{-\alpha(\tau - t)} \|x\|_2$$

where $\phi(\tau, t, x)$ denotes solution at time $\tau$ starting from $(t, x)$.

(ii) There exists a function $V(t,x)$ and constants $h, \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$ s.t. $\forall\ x \in B_h(0)$, $t \geq 0$,

$$\alpha_1 \|x\|_2^2 \leq V(t,x) \leq \alpha_2 \|x\|_2^2$$

$$\left( \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f \right) \leq -\alpha_3 \|x\|_2^2$$

$$\left\| \frac{\partial V}{\partial x} \right\|_2 \leq -\alpha_4 \|x\|_2^2$$

<u>Proof</u>

(i) $\Rightarrow$ (ii)

define
$$V(t,x) \triangleq \int_t^{t+T} \|\phi(\tau,t,x)\|_2^2 \, d\tau$$
$$\leq \int_t^{t+T} m^2 e^{-2\alpha(\tau-t)} \|x\|_2^2 \, d\tau$$
$$= \frac{m^2}{2\alpha}\left(1-e^{-2\alpha T}\right)\|x\|_2^2$$
$$\triangleq \alpha_2 \|x\|_2^2$$

Letting $L$ be the Lipschitz constant on $B_h(0)$,
$$\|x\|_2 \, e^{-L(\tau-t)} \leq \|\phi(\tau,t,x)\|_2$$

and hence
$$V(t,x) \geq \int_t^{t+T} \|x\|_2^2 \, e^{-2L(\tau-t)} \, d\tau$$
$$= \frac{(1-e^{-2LT})}{2L}\|x\|_2^2$$
$$\triangleq \alpha_1 \|x\|_2^2$$

$$\frac{dV}{dt}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\cdot f$$
$$= \|\phi(t+T,t,x)\|_2^2 - \overbrace{\|\phi(t,t,x)\|_2^2}^{=\ \|x\|_2^2}$$
$$+ \int_t^{t+T} \frac{d}{dt}\|\phi(\tau,t,\underset{x(t)}{x})\|_2^2 \, d\tau$$

But, $\phi(\tau, t+\Delta t, x(t+\Delta t)) = \phi(\tau,t,x(t))$.
so the 3rd time <u>vanishes</u>!

$$\leq \|x\|_2^2 m^2 e^{-2\alpha T} - \|x\|_2^2$$

$$= -\|x\|_2^2 (1 - m^2 e^{-2\alpha T})$$

$$\triangleq -\|x\|_2^2 \alpha_3$$

So pick $T > \frac{1}{\alpha} \ln(m)$. Then

$\alpha_3 > 0$.

$$\cancel{\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \partial f}$$

$$\frac{\partial v}{\partial x_i} = \int_t^{t+T} 2 \sum_{j=1}^{n} \phi_j(\tau, t, x) \frac{\partial \phi_j}{\partial x_i}(\tau, t, x) \, d\tau$$

Let $Q_{ij}(\tau, t, x) \triangleq \dfrac{\partial \phi_j(\tau, t, x)}{\partial x_i}$

& $A_{ij}(t, x) \triangleq \dfrac{\partial f_i(t,x)}{\partial x_j}$

$$\frac{d}{d\tau} Q(\tau, t, x) = \frac{d}{d\tau} \frac{\partial \phi}{\partial x}(\tau, t, x)$$

$$= \frac{\partial}{\partial x} \frac{d}{d\tau} \phi(\tau, t, x) \qquad \text{switching order of differential}$$

$$= \frac{\partial}{\partial x} f(\tau, \phi(\tau, t, x)) \qquad \text{def}^{n} \text{ of } \phi$$

$$\cancel{= A(\tau, x) Q(\tau, t, x)}$$

$$= A(\tau, \phi(\tau, t, x)) Q(\tau, t, x) \qquad \text{chain rule}$$

The assumption that partials (with respect to $x$) of $f$ are continuous and bounded on $\bar{B}_h(0)$

$\Rightarrow \quad \| A(\cdot,\cdot) \|_2 \le k$ for some $k$. on $B_h(0)$.

$$\Rightarrow \quad \| Q(\tau, t, x) \|_2 \le e^{k(\tau - t)}$$

Hence
$$\left\| \frac{\partial V}{\partial x} \right\|_2 \le 2 \int_t^{t+T} m \|x\|_2 \, e^{(k-\alpha)(\tau - t)} \, d\tau$$

$$\triangleq \alpha_4 \|x\|_2$$

( check that $V$ is defined only $B_{h'}(0)$ $\quad h' \le \frac{h}{m}$ )

Proof of (ii) $\Rightarrow$ (i)

$$\dot{V} \le -\alpha_3 \|x\|_2^2 \le -\frac{\alpha_3}{\alpha_2} V$$

$$\Rightarrow \alpha_1 \|x(t)\|_2^2 \le V(t, x(t)) \le V(t_0, x(t_0)) \, e^{-\frac{\alpha_3}{\alpha_2}(t - t_0)}$$

$$\le \alpha_2 \|x(t_0)\|_2^2 \, e^{-\frac{\alpha_3}{\alpha_2}(t - t_0)}$$

$$\Rightarrow \quad \|x(t)\|_2^2 \le \frac{\alpha_2}{\alpha_1} \|x(t_0)\|_2^2 \, e^{-\frac{\alpha_3}{\alpha_2}(t - t_0)}$$

$$\Rightarrow \quad \|x(t)\|_2 \le \sqrt{\frac{\alpha_2}{\alpha_1}} \|x(t_0)\|_2 \, e^{-\frac{\alpha_3}{2\alpha_2}(t - t_0)}$$

So we can take $\alpha = \frac{\alpha_3}{2\alpha_2}$

$$m = \sqrt{\frac{\alpha_2}{\alpha_1}}$$