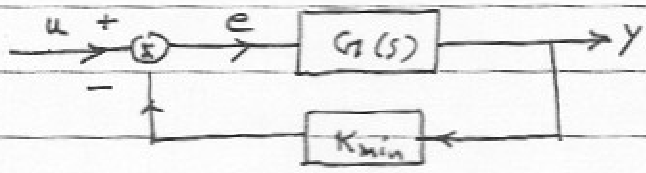
PSK (04/30/00)

The transfer function $\tilde{G}(s)$ of the system

$$\dot{x} = (A - BK_{min}C)x + Bu$$

$$y = Cx$$

is the same as the transfer function of the closed loop system in the ~~adjoinig~~ block diagram



where $G(s) = C(s\mathbb{1} - A)^{-1}B$ as before.

Observe that, in terms of Laplace transforms of inputs and outputs,

$$Y(s) = G(s)\,E(s)$$
$$E(s) = U(s) - K_{min}Y(s)$$

$$\Rightarrow \quad Y(s) = G(s)\,U(s) - G(s)K_{min}Y(s)$$

$$\Rightarrow \quad (\mathbb{1} + G(s)K_{min})Y(s) = G(s)U(s)$$

$$\Rightarrow \quad Y(s) = (\mathbb{1} + G(s)K_{min})^{-1}G(s)U(s)$$

$$\Rightarrow \quad \tilde{G}(s) = (\mathbb{1} + G(s)K_{min})^{-1}G(s).$$

On the other hand

$$E(s) = U(s) - K_{min}Y(s)$$
$$= U(s) - K_{min}G(s)E(s)$$

$$\Rightarrow \quad (\mathbb{1} + K_{min} G(s)) E(s) = U(s)$$

$$\Rightarrow \quad E(s) = (\mathbb{1} + K_{min} G(s))^{-1} U(s)$$

$$\Rightarrow \quad Y(s) = G(s) E(s)$$
$$= G(s) (\mathbb{1} + K_{min} G(s))^{-1} U(s)$$

So we have two formulas

$$\tilde{G}(s) = (\mathbb{1} + G(s) K_{min})^{-1} G(s)$$

$$= G(s) (\mathbb{1} + K_{min} G(s))^{-1}$$

(and of course ~~they~~ are equivalent as a little algebra shows).

Now $A - B K_{min} C$ is Hurwitz iff all the poles of $\tilde{G}(s) = G(s)(\mathbb{1} + K_{min} G(s))^{-1}$ are in $\mathbb{C}^-$.

Applying the sector condition to $\tilde{\psi}$,

$$\tilde{\psi}(t,y)^T (\tilde{\psi}(t,y) - K y) \le 0$$

$$\Leftrightarrow \quad (\psi(t,y) - K_{min} y)^T (\psi(t,y) - (K_{min} + K) y) \le 0$$

$$\Leftrightarrow \quad (\psi(t,y) - K_{min} y)^T (\psi(t,y) - K_{max} y) \le 0$$

for $K_{max} = K_{min} + K$.

PSK
(04/30/00)

The relevant positive real transfer function is

$$\tilde{Z}(s) = \mathbb{1} + K\,\tilde{G}(s)$$

$$= \mathbb{1} + KG\,(\mathbb{1} + K_{min}G)^{-1}$$

$$= (\mathbb{1} + K_{min}G)(\mathbb{1} + K_{min}G)^{-1} + KG(\mathbb{1} + K_{min}G)^{-1}$$

$$= (\mathbb{1} + (K_{min} + K)G)\,(\mathbb{1} + K_{min}G)^{-1}$$

$$= (\mathbb{1} + K_{max}G)\,(\mathbb{1} + K_{min}G)^{-1}$$

Now we are ready to state the multivariable circle criterion <u>without</u> the Hurwitz Assumption on A

<u>Theorem</u> (CIRCLE CRITERION) Let $[A, B, C]$ be a controllable and observable triple. Suppose ψ satisfies the sector condition

$$(\psi^{T}(t,y) - K_{min}\,y)^T(\psi(t,y) - K_{max}\,y) \leq 0$$

$\forall\ t \geq 0$, $\forall\ y \in \mathbb{R}^m$, and $K = K_{max} - K_{min} = K^T > 0$ given.

Then the closed loop system is absolutely stable <u>provided</u>

(a) $\tilde{G}(s) = G(s)\,(\mathbb{1} + K_{min}G(s))^{-1}$ is "Hurwitz" (analytic in $\{s : Re(s) \geq 0\}$)

(b) $\tilde{Z}(s) = (\mathbb{1} + K_{max}G)(\mathbb{1} + K_{min}G)^{-1}$ is strict positive real.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -\psi(t,y)$$

Proof : From the remarks preceding the above statement, it is clear that all one has to do is to appeal to the equivalences of closed loop systems with and without the loop transformation arising from the feedback $A \mapsto A - BK_{min}C$ and appeal to the Hurwitz case already proved. $\blacksquare$

Where does this name "circle criterion" come from? This is an interesting story going back to the work of ~~Ny~~ Harry Nyquist the AT&T Mathematician who investigated graphical methods for feedback amplifier stability in long-distance (transatlantic) telephony. This is a direct application of the principle of the argument in complex variable theory.

This work of Nyquist was in the 1930's. He is the same man who developed the theoretical understanding of thermal noise (Nyquist-Johnson noise) in electrical devices in parallel with the work of the experimentalist Johnson (in 1928). Extremely important to day in the context of MEMS devices

First specialize to the single input single output case.

Let $\Pi_G \triangleq \{ u + jv = G(j\omega) \mid \omega \in \mathbb{R} \}$
$= $ image of the imaginary axis under $G(\cdot)$

be the Nyquist locus of G.

Theorem (Nyquist) Let Π_G be bounded (i.e. G is proper and no poles on the $j\omega$ axis). We will say that Π_G encircles

$G(s) = \frac{q(s)}{p(s)}$ is proper if $\deg q(s) \leq \deg p(s)$
strictly proper if ... $<$

a point $u_0 + jv_0$, ρ times, if $u_0 + jv_0$ is not on Γ_G and $2\pi\rho$ = net <u>increase</u> in the argument of $G(j\omega) - (u_0 + jv_0)$ as ω increases from $-\infty$ to $+\infty$.

clockwise encirclement $\Leftrightarrow$ direction of increasing argument

counterclockwise encirclement $\Leftrightarrow$ direction of decreasing argument.

Suppose Γ_G is bounded. If G has ν poles in the half plane $\mathbb{C}^+$ then $\frac{G}{1+kG}$ has $\rho + \nu$ poles in the half plane $\mathbb{C}^+$ if the point $-\frac{1}{k} + j0$ is not on Γ_G <u>and</u> Γ_G encircles it ρ times in the clockwise sense.

(proof: see Franklin et. al.) reference list for this class.

<u>Corollary</u> If Γ_G is bounded and $-\frac{1}{k} + j0$ is not on Γ_G and G has ν poles in $\mathbb{C}^+$ then the feedback $u = -ky$ stabilizes the closed loop system if Γ_G encircles $-\frac{1}{k} + j0$ ν times in the <u>counterclockwise</u> direction

Lemma Let $g(s)$ be a scalar transfer function. Let $g(s)$ be proper (i.e. $g(s) = \frac{q(s)}{p(s)} + d$ where $\deg(q) < \deg(p)$, p monic and d a constant). poles of $g(s)$ all lie in $\mathbb{C}^-$. Then $g(s)$ is strict positive real iff

$$\mathrm{Re}\,(g(j\omega)) > 0 \quad \forall\, \omega \in \mathbb{R}.$$

Proof : < See H. Khalil → page 404.

Theorem: Let $g(s)$ be a scalar transfer function $= c(sI-A)^{-1}b$, $[A, b, c]$ controllable and observable. Let $\psi(t,y)$ satisfy the sector condition:

$$\alpha y^2 \leq y\psi(t,y) \leq \beta y^2$$

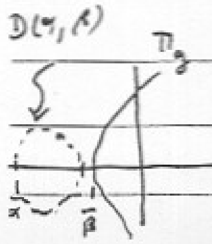
Then absolute stability of the closed loop system

$$\dot{x} = Ax + bu$$

$$y = cx$$

$$u = -\psi(t,y)$$

holds provided one of the following conditions apply



(i) If $0 < \alpha < \beta$, the Nyquist locus does not enter the disk $D(\alpha, \beta)$ and encircles it ν times in the counterclockwise direction where $\nu = \#$ poles of $g(s)$ in $\mathbb{C}^+$

(ii) If $0 = \alpha < \beta$, $g(s)$ is "Hurwitz", and the Nyquist plot Γ_g lies to the right of the line $Re(s) = -\frac{1}{\beta}$

(iii) If $\alpha < 0 < \beta$, $g(s)$ is "Hurwitz" and the Nyquist plot Γ_g lies in the interior of the disk $D(\alpha, \beta)$

<u>Proof</u> : specializing the multivariable circle criterion to this case, we seek conditions to ensure that

(a) $\dfrac{g(s)}{1+\alpha g(s)}$ is Hurwitz <u>and</u>

(b) $\dfrac{1+\beta g(s)}{1+\alpha g(s)}$ is strict positive real.

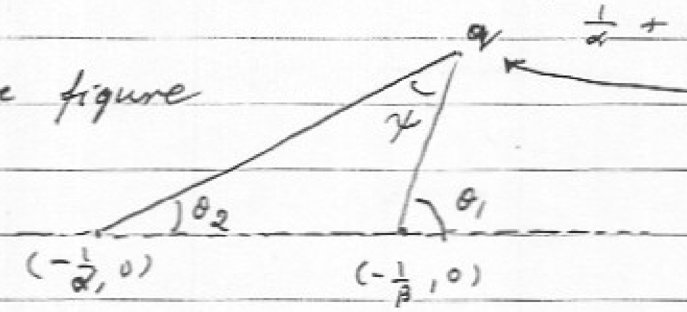
For (b) it is equivalent to check

$$Re\left(\frac{1+\beta g(j\omega)}{1+\alpha g(j\omega)}\right) > 0 \quad \forall \omega \in \mathbb{R}$$

<u>In case (i)</u> $0 < \alpha < \beta$, this is equivalent to checking

$$Re\left(\frac{\frac{1}{\beta} + g(j\omega)}{\frac{1}{\alpha} + g(j\omega)}\right) > 0 \quad \forall \omega \in \mathbb{R}$$

Consider the figure



$q \in \Gamma_g$

$\psi = \theta_1 - \theta_2$

$\theta_1 = \arg\left(\frac{1}{\beta} + g(j\omega)\right)$

$\theta_2 = \arg\left(\frac{1}{\alpha} + g(j\omega)\right)$

$$\mathrm{Re}\left(\frac{1/\beta + g(j\omega)}{1/\alpha + g(j\omega)}\right) = r \cdot \cos\psi$$

where $r \geq 0$. $\cos\psi > 0 \Leftrightarrow \psi = \theta_1 - \theta_2 < \frac{\pi}{2}$

By elementary geometry g has to lie outside $D(\alpha, \beta)$ the disk with diameter joining $(-\frac{1}{\alpha}, 0)$ and $(-\frac{1}{\beta}, 0)$. For the encirclement condition use the corollary to Nyquist.

In case (ii) the condition for strict positive reality is $\mathrm{Re}\left(\frac{1}{\beta} + g(j\omega)\right) > 0$

$\leftrightarrow \cos(\theta_1) > 0$

$\leftrightarrow \theta_1 < \pi/2$

$\leftrightarrow$ Γ_g lies to the right of the vertical line $\{s : \mathrm{Re}(s) = -\frac{1}{\beta}\}$.

In case (iii) same arguments as in (i) above but we seek $\psi > \frac{\pi}{2} \leftrightarrow \Gamma_g$ lies in the interior of the disk $D(\alpha, \beta)$.

⟨because α and β have opposite sign the strict positive reality condition is

$$\mathrm{Re}\left(\frac{\frac{1}{\beta} + g(j\omega)}{\frac{1}{\alpha} + g(j\omega)}\right) < 0$$

⟩

$\boxtimes$

The Popov criterion for absolute stability is based on

(i) restrictions on ψ

(ii) use of non-quadratic Lyapunov function

(i) $\psi = \psi(y) = (\psi_1(y_1), \psi_2(y_2), \ldots, \psi_m(y_m))^T$

$\psi^T(y)(\psi(y) - Ky) \leq 0 \qquad K = \mathrm{diag}(\beta_1, \ldots, \beta_m), \ \beta_i \geq 0 \ \forall i$

(ii) $V(x) = x^T P x + 2\eta \int_0^y \sum_{i=1}^m \psi_i(\sigma_i)\beta_i \, d\sigma_i$

$\triangleq x^T P x + 2\eta \int_0^y \psi^T(\sigma) K \, d\sigma \qquad$ here $y = Cx$

By sector condition, $\psi(\sigma) \geq 0$ for $\sigma \geq 0$ (componentwise) $\Rightarrow \int_0^y \sum_1^m \psi_i(\sigma_i)\beta_i \, d\sigma_i \geq 0$

$\Rightarrow V(x) > 0$ (as long as $P = P^T > 0$).

Along trajectories of the (usual) closed loop system

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} + 2\eta \psi^T K \dot{y} \qquad (y = Cx)$$

$$= (Ax - B\psi)^T P x + x^T P (A - B\psi) + 2\eta \psi^T K C (Ax - B\psi)$$

$$= x^T (A^T P + PA) x - 2 x^T P B \psi + 2\eta \psi^T K C (Ax - B\psi)$$

Since $-2\psi^T(\psi - Ky) \geq 0$ we get

by the sector condition

$$\dot{V} \leq x^T (A^T P + PA) x - 2 x^T P B \psi + 2\eta \psi^T K C (Ax - B\psi) - 2\psi^T(\psi - Ky)$$

$$= x^T (A^T P + PA) x - 2 x^T (PB - \eta A^T C^T K - C^T K)\psi - \psi^T (2\mathbb{1} + \eta KCB + \eta B^T C^T K)\psi$$

Choose η small enough s.t.

$$2\mathbb{1} + \eta KCB + \eta B^T C^T K > 0$$

$\longleftrightarrow$ we can find W s.t.

$$W^T W = 2\mathbb{1} + \eta KCB + \eta B^T C^T K$$
$$= (\mathbb{1} + \eta KCB) + (\quad)^T$$

Suppose $P = P^T > 0$ and $\exists\, L$ and $\epsilon > 0$ s.t.

$$A^T P + PA = -L^T L - \varepsilon P$$
$$PB = C^T K + \eta A^T C^T K - L^T W$$
$$= (C + \eta CA)^T K - L^T W$$

Then

$$\dot{V}(x) \le -\varepsilon x^T P x - x^T L^T L x + 2x^T L^T W \psi - \psi^T W^T W \psi$$
$$= -\varepsilon x^T P x - (Lx - W\psi)^T (Lx - W\psi)$$
$$\le -\varepsilon x^T P x$$
$$< 0 \qquad x \neq 0.$$

Thus we get absolute stability. The question of P, L, ε, W is settled by the KYP lemma.

$$Z(s) = (\mathbb{1} + \eta KCB) + (KC + \eta KCA)(s\mathbb{1} - A)^{-1} B$$
$$= \mathbb{1} + \eta KC(s\mathbb{1} - A + A)(s\mathbb{1} - A)^{-1} B + KC(s\mathbb{1} - A)^{-1} B$$
$$= \mathbb{1} + \eta s KC(s\mathbb{1} - A)^{-1} B + KC(s\mathbb{1} - A)^{-1} B$$
$$= \mathbb{1} + (\eta s + 1) K G(s)$$

Suppose η is chosen that $-\frac{1}{\eta}$ is not an eigenvalue of A. Then $(A, K(C+\eta CA))$ is observable iff (A, C) is observable.

Then by KYP, P, L, ε, W exist if

$$Z(s) = \mathbb{1} + (\eta s + 1) K G(s)$$ is strict positive real. We have proved

THEOREM (multivariable Popov criterion)

$$(*) \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(y) \end{aligned}$$

A Hurwitz, (A, B, C) $\begin{Bmatrix} \text{controllable } \& \\ \text{observable} \end{Bmatrix}$

$\psi(y) = (\psi_1(y_1), \ldots, \psi_m(y_m))^T$

$K = \mathrm{diag}(\beta_1, \ldots, \beta_m)$

$\beta_i > 0 \quad \forall i.$

$$0 \le y_i \psi_i(y_i) \le \beta_i y_i^2 \longleftarrow \text{(sector condition)}$$

Then $(*)$ is absolutely stable if $\exists\, \eta \geq 0$ s.t. $-\frac{1}{\eta} \in \text{spectrum}(A)$ and

$$Z(s) = \mathbb{1}_m + (1 + \eta s) K G(s)$$

is strict positive real. ▨

Choose η small enough s.t.

$$2\mathbb{1} + \eta KCB + \eta B^T C^T K \geq 0$$

$\leftrightarrow$ we can find W s.t.

$$W^T W = 2\mathbb{1} + \eta KCB + \eta B^T C^T K$$
$$= (\mathbb{1} + \eta KCB) + (\qquad)^T$$

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$$A^T P + PA = -L^T L - \varepsilon P$$
$$PB = C^T K + \eta A^T C^T K - L^T W$$
$$= (C + \eta CA)^T K - L^T W$$

Then

$$\dot V(x) \leq -\varepsilon x^T P x - x^T L^T L x + 2x^T L^T W \psi - \psi^T W^T W \psi$$
$$= -\varepsilon x^T P x - (Lx - W\psi)^T (Lx - W\psi)$$

$$\leq -\varepsilon x^T P x$$
$$< 0 \qquad x \neq 0 .$$

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THEOREM (multivariable Popov criterion)

$$(*) \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(y) \end{aligned}$$

A Hurwitz, (A, B, C) $\left\{\begin{matrix} \text{controllable \&} \\ \text{observable} \end{matrix}\right\}$

$\psi(y) = (\psi_1(y_1), \dots, \psi_m(y_m))^T$

$K = \operatorname{diag}(\beta_1, \dots, \beta_m)$

$\beta_i > 0 \quad \forall i.$

$0 \le y_i \psi_i(y_i) \le \beta_i y_i^2 \longleftarrow$ (sector condition)

Then $(*)$ is absolutely stable if $\exists\, \eta \geq 0$ s.t. $-\frac{1}{\eta} \in$ spectrum(A) and

$$Z(s) = \mathbb{1}_m + (1 + \eta s) K G(s)$$

is strict positive real. $\blacksquare$

Remark (a) With $\eta = 0$, this reduces to a special case of the circle criterion

(b) With $\eta > 0$, we get absolute stability under weaker conditions (but for a restricted class of nonlinear maps ψ)

(c) For $m = 1$ (SISO case), we have a graphical test

choose η s.t. $Z(\infty) = \lim_{s \to \infty} Z(s) = W^2 > 0$.

Then $Z(s)$ is strict positive real iff

$$\mathrm{Re}\left[1 + (1 + \eta j\omega)\, k\, g(j\omega)\right] > 0 \quad \forall \omega \in \mathbb{R}$$

note $k > 0$

$$\Longleftrightarrow \quad \frac{1}{k} + \mathrm{Re}(g(j\omega)) - \eta\, \omega\, \mathrm{Im}(g(j\omega)) > 0 \quad \forall \omega \in \mathbb{R}$$

$\Longleftrightarrow$ Popov locus/plot lies to the right of the line that intercepts the point $-\frac{1}{k} + j0$ with slope η

Here: Popov locus $= \mathcal{P}_g = \left\{ u + jv : \begin{array}{l} u = \mathrm{Re}(g(j\omega)) \\ v = \omega \mathrm{Im}(g(j\omega)) \end{array} \right\}$ $\forall \omega \in \mathbb{R}$