ENEE 661 Nonlinear Control Systems

Lecture 6 (part iii)

Here we wish to state and prove a theorem on assessing stability of nonlinear systems via linearization. We need some background.

Fundamental Theorem of Integral Calculus

Let X and Y be two finite dimensional vector spaces. Let $U \overset{\text{open}}{\subset} X$ and let $f : U \to Y$ be C^1.

If $x + ty \in U \quad \forall \quad t \in [0,1]$

(e.g. if $U = \overset{\circ}{B}_r(x)$)

then

$$f(x+y) = f(x) + \int_0^1 Df(x+ty)\,y\,dt$$

[note: $Df(z)h = \frac{d}{ds} f(z+sh)\Big|_{s=0}$ the Frechet derivative]

Proof Let $g(t) = f(x+ty) \qquad 0 \le t \le 1.$

For $0 < t < 1$, by chain rule

$$g'(t) = Df(x+ty)y$$

Let $h(t) = f(x) + \int_0^t Df(x+sy)y\,ds$

$$0 \le t \le 1$$

$$h'(t) = Df(x+ty)y \qquad 0 < t < 1.$$

Hence $g'(t) = h'(t) \qquad 0 < t < 1$

$g(t) = h(t) + \text{constant} \qquad 0 \le t \le 1$

But $g(0) = f(x) = \ell(0)$.

PSK
04/12/00

So $g(1) = \ell(1)$ ▨

Remark: The theorem is true as stated when X and Y are general Banach spaces. But then we need to have a suitable theory of the integral. In finite dimensions we are content with the Riemann integral. ▨

We can rewrite

$$f(x) = f(0) + \int_0^1 Df(tx)\, x \, dt$$
$$= f(0) + M(x)\, x$$

where $M(x) = \int_0^1 Df(tx)\, dt$ (an x dependent matrix-valued function).

Consider a C^1 vector field $f(x)$ with $f(0) = 0$. Let $A = \left(\frac{\partial f}{\partial x}\right)\Big|_0 = (Df)(0)$.

Let us write $f(x) = Ax + f(x) - Ax$
$$= Ax + g(x)$$

where $g(x) \triangleq f(x) - Ax$ (different from g on previous page)

$g(\cdot)$ is C^1, since f is C^1.

Applying the fundamental theorem of integral calculus one can write,

PSK 04/12/00

$$g(x) = g(0) + N(x)\,x$$

where $g(0) = f(0) - A\cdot 0 = 0$, and

$$N(x) = \int_0^1 Dg(tx)\,dt$$

$$= \int_0^1 (Df(tx) - A)\,dt$$

$$\lim_{x\to 0} N(x) = \int_0^1 (\lim_{x \to 0} Df(tx) - A)\,dt$$

$$= \int_0^1 (Df(0) - A)\,dt$$

$$= \int_0^1 (A - A)\,dt = 0$$

Thus $\dfrac{\|g(x)\|}{\|x\|} \le \|N(x)\| \to 0$ as $\|x\| \to 0$, in any norm.

Then, for any $\gamma > 0$ (arbitrarily small), there exists an $r > 0$ such that

$$\|g(x)\|_2 < \gamma\, \|x\|_2 \quad \forall\, \|x\|_2 < r$$

This key property will be used below. We will also stick to the notation above for $f(x)$, $g(x)$, $N(x)$ etc.

Remark

If $f = f(t,x)$ and $f(t,0) \equiv 0$,

then $g(t,x) = f(t,x) - A(t)x$

where $A(t) = \left.\dfrac{\partial f}{\partial x}(t,x)\right|_{x=0}$

has the property

$$\frac{\|g(t,x)\|_2}{\|x\|_2} \to 0 \quad \text{as } \|x\|_2 \to 0$$

for each $t \geq 0$.

But this property does not hold uniformly in general i.e. one cannot take for granted that

(uniform order condition)

$$\lim_{\|x\|_2 \to 0} \sup_{t \geq 0} \frac{\|g(t,x)\|_2}{\|x\|_2} = 0$$

e.g: $\dot{x} = f(t,x) = -x + t x^2$

Such a uniform hypothesis is needed for a linearization based stability theorem for time varying nonlinear systems

See Sastry page 214–215

→ TIME INVARIANT CASE

PSK 04/12/00

<u>Theorem</u> 6 (Stability via linearization)

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Assume f is C^1 on a neighborhood $B_\rho(0)$ of 0. Let

$$A = \left(\frac{\partial f}{\partial x}\right)\Big|_{x=0}.$$

If spectrum $(A) \subseteq \mathbb{C}^-$ the open l-h-p. then the origin is an asymptotically stable equilibrium point of the <u>nonlinear</u> system.

<u>Proof</u> Let $Q = Q^T > 0$. Then $\exists!\ P > 0$ such that

$$A^T P + PA = -Q$$

(proof: A is Hurwitz and hence

$$P = \int_0^\infty e^{A^T\sigma} Q e^{A\sigma} d\sigma$$

is a convergent integral and is positive definite).

Let $v(x) = x^T P x$ and compute along trajectories of $\dot{x} = f(x)$, the derivative

$$\dot{v} = \dot{x}^T P x + x^T P \dot{x}$$

$$= (Ax + g(x))^T P x + x^T P (Ax + g(x))$$

$$= x^T(A^TP + PA)x \quad + 2x^TPg(x)$$
$$= -x^TQx \quad + 2x^TPg(x)$$

But $x^TPg(x) \leqq \|x\|_2 \, \|Pg(x)\|_2$ (Cauchy Schwartz)

$$\leq \|x\|_2 \, \|P\|_2 \, \|g(x)\|_2$$
$$< \gamma \|x\|_2 \, \|P\|_2 \, \|x\|_2 \qquad (\text{if } \|x\|_2 < r < \rho) \qquad (1)$$

On the other hand

$$0 < \lambda_{min}(Q) \|x\|_2^2 < x^TQx < \lambda_{max}(Q) \|x\|_2^2.$$
$$\Rightarrow \quad -x^TQx < -\lambda_{min}(Q) \|x\|_2^2. \qquad (2)$$

Taken together, inequalities (1) and (2) imply that

$$\dot{V} < -\lambda_{min}(Q) \|x\|_2^2 + 2\gamma \|P\|_2 \|x\|_2^2$$
$$= (-\lambda_{min}(Q) + 2\gamma \|P\|_2) \|x\|_2^2.$$

Picking Q, determines $\lambda_{min}(Q)$ and $\|P\|_2$. So we can pick r sufficiently small so that γ is sufficiently that,

$$-\lambda_{min}(Q) + 2\gamma \|P\|_2 < 0.$$

By Lyapunov, we have asymptotic stability.

⟨Of course, a smaller r means the (estimate of the) domain of attract $B_r(0)$ is smaller⟩

Theorem 7

(Instability)

PSK
04/12/00

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Assume f is C^1 on a neighborhood $B_\rho(0)$ of 0. Let $A = \left(\frac{\partial f}{\partial x}\right)\Big|_0$.

If spectrum $A \subseteq \mathbb{C}^+$ the open r.h.p then the origin is an ~~asympt~~ unstable equilibrium point of the nonlinear system.

Proof spec$(-A) \subseteq \mathbb{C}^-$

So for $Q = Q^T > 0$, $\exists!$ $P > 0$ such that

$$(-A^T)P + P(-A) = -Q. \qquad \text{———} \quad (1)$$

Along trajectories of $\dot{x} = f(x) = Ax + g(x)$ the derivative of $V(x) = x^T P x$ satisfies

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x}$$

$$= x^T(A^T P + PA)x + 2x^T P g(x)$$

$$= x^T Q x + 2x^T P g(x)$$

$$\geq \lambda_{min}(Q)\|x\|_2^2 - 2|x \cdot P g(x)|$$

$$\geq \lambda_{min}(Q)\|x\|_2^2 - 2\|x\|_2 \cdot \|P\|_2 \|g(x)\|_2$$

$$> \lambda_{min}(Q)\|x\|_2^2 - 2\|x\|_2 \cdot \|x\|_2 \gamma \|P\|_2$$

(for $\|x\|_2 < r < \rho$, r sufficiently small).

Pick Q. this determines P and $\|P\|_2$; pick r sufficiently small so that γ is sufficiently small so that

$$\dot{V} > (\lambda_{min}(Q) - 2\gamma \|P\|_2) \|x\|_2^2$$

$$> 0 \qquad \forall\, x \in B_r(0) - \{0\}.$$

By Lyapunov's instability theorem, it follows that 0 is unstable for the nonlinear system ▨

The hypotheses of theorem 7 are too strong. we can do better.

<u>Theorem 8</u> In the statement of theorem 7 assume that <u>at least</u> one of the eigenvalues of A is in $\mathbb{C}^+$. Then 0 is unstable □

<u>Proof</u> In general A has a splitting of spectrum

$$\text{spec}(A) = \sigma_- \cup \sigma_0 \cup \sigma_+$$

where $\sigma_- \subseteq \mathbb{C}^-$, $\sigma_0 \subseteq j\omega$ axis and $\sigma_+ \subset \mathbb{C}^+$ and we have assumed that $\sigma_+ \neq \phi$.

Then there exists $\varepsilon > 0$ such that

$$\text{spec}(A - \tfrac{\varepsilon}{2}\mathbb{1}) = \sigma_-^{\varepsilon} \cup \sigma_+^{\varepsilon}$$

where $\sigma_-^{\varepsilon} \subseteq \mathbb{C}^-$ and $\sigma_+^{\varepsilon} \subseteq \mathbb{C}^+$

and $\sigma_+^{\varepsilon} \neq \phi$. (we got rid of pure imaginary eigenvalues by a ~~left~~ right shift of the imaginary axis)

Let $A^{\varepsilon} \triangleq A - \frac{\varepsilon}{2} \mathbb{1}$

There is an nonsingular, real matrix T (recall real Jordan form) such that

$$T A^{\varepsilon} T^{-1} = T A T^{-1} - \frac{\varepsilon}{2} \mathbb{1}$$

$$= \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right) - \frac{\varepsilon}{2} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$= \left(\begin{array}{c|c} A_1 - \frac{\varepsilon}{2} \mathbb{1} & 0 \\ \hline 0 & A_2 - \frac{\varepsilon}{2} \mathbb{1} \end{array} \right)$$

where $\operatorname{spec}\left(A_1 - \frac{\varepsilon}{2}\mathbb{1}\right) \subseteq \mathbb{C}^-$ and $\operatorname{spec}\left(A_2 - \frac{\varepsilon}{2}\mathbb{1}\right) \subseteq \mathbb{C}^+$.

Let $Q_i = Q_i^T > 0$ and let $P_i = P_i^T > 0$ be the unique matrices satisfying

$$\left(A_1 - \frac{\varepsilon}{2}\mathbb{1}\right)^T P_1 + P_1 \left(A_1 - \frac{\varepsilon}{2}\mathbb{1}\right) = -Q_1$$

$$-\left(A_2 - \frac{\varepsilon}{2}\mathbb{1}\right)^T P_2 - P_2 \left(A_2 - \frac{\varepsilon}{2}\mathbb{1}\right) = -Q_2$$

(we have used the fact that $\operatorname{spec}\left(-\left(A_2 - \frac{\varepsilon}{2}\mathbb{1}\right)\right) \subseteq \mathbb{C}^-$)

Consider $z = Tx$. Then $\dot{z} = T\dot{x} = Tf(x)$

$$\dot{z} = T\left(Ax + g(x)\right)$$

where $g(x) \triangleq f(x) - Ax$.

$$\Rightarrow \dot{z} = T A T^{-1} z + T g\left(T^{-1} z\right)$$

$$= \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array} \right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} h_1(z) \\ h_2(z) \end{pmatrix}$$

$\therefore$ $\because$ By hypotheses / defn of $g(\cdot)$ it follows that

$h(0) = 0$ and given $\gamma > 0$, $\exists\ r > 0$ s.t.

$\|h(z)\|_2 < \gamma \|z\|_2 \quad \forall\ \|z\|_2 < r$.

9

PSK 04/17/00

Define $\quad V(z) = -z_1^T P_1 z_1 + z_2^T P_2 z_2$.

Then
$$\dot{V}(z) = -\dot{z}_1^T P_1 z_1 - z_1^T P_1 \dot{z}_1 + \dot{z}_2^T P_2 z_2 + z_2^T P_2 \dot{z}_2$$

$$= -z_1^T (A_1^T P_1 + P_1 A_1) z_1 - 2 z_1^T P_1 h_1(z) + z_2^T (A_2^T P_1 + P_2 A_2) z_2 + 2 z_2^T P_2 h_2(z)$$

$$= -z_1^T \left((A_1 - \tfrac{\varepsilon}{2}\mathbb{1})^T P_1 + P_1 (A_1 - \tfrac{\varepsilon}{2}\mathbb{1}) \right) z_1 - \varepsilon z_1^T P_1 z_1 - 2 z_1^T P_1 h_1(z)$$

$$+ z_2^T \left((A_2 - \tfrac{\varepsilon}{2}\mathbb{1})^T P_2 + P_2 (A_2 - \tfrac{\varepsilon}{2}\mathbb{1}) \right) z_2 + \varepsilon z_2^T P_2 z_2 + 2 z_2^T P_2 h_2(z)$$

$$= + z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + \varepsilon V(z) - 2 z_1^T P_1 h_1 + 2 z_2^T P_2 h_2$$

$$= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + \varepsilon V(z) - 2 z^T \begin{pmatrix} P_1 h_1 \\ -P_2 h_2 \end{pmatrix}$$

$$\geq \lambda_{min}(Q_1) \|z_1\|_2^2 + \lambda_{min}(Q_2) \|z_2\|_2^2 + \varepsilon V(z) - 2 \|z\|_2 \cdot \max(\|P_1\|_2, \|P_2\|_2)\, \|h(z)\|_2^2$$

$$\geq (\alpha - 2\sqrt{2}\, \gamma \beta) \|z\|_2^2 + \varepsilon V(z)$$

where $\alpha = \min(\lambda_{min}(Q_1), \lambda_{min}(Q_2))$

$\beta = \max(\|P_1\|_2, \|P_2\|_2)$

$\forall \|z\|_2 < r$

and $\gamma > 0$

Let $U = \{ z \in B_r(0) \mid V(z) > 0 \}$

Then $\dot{V} > 0$ on U — in fact there is a quadratic class $\mathcal{K}$ function bounding $\dot{V}$ below —, provided $\gamma < \dfrac{\alpha}{2\sqrt{2}\beta}$

By Cetaev's instability theorem, 0 is unstable. $\square$

<u>Remark</u>: The cases where there are no eigenvalues on the open right half plane but there are eigenvalues on the imaginary axis, are called critical cases — one cannot say anything about stability via linearization.

SAVE

$$\lim_{t\to 0} (Df)(tx)h = \lim_{t\to 0} \frac{d}{ds} f(tx+sh)\Big|_{s=0}$$

$$= \lim_{t\to 0}\lim_{s\to 0} \frac{f(tx+sh)-f(tx)}{s}$$

$$= \lim_{s\to 0}\lim_{t\to 0} \frac{f(tx+sh)-f(tx)}{s}$$

$$= \lim_{s\to 0} \frac{f(sh)-f(0)}{s}$$

$$= \frac{d}{ds} f(sh)\Big|_{s=0}$$

$$= (Df)(0)h$$