Lecture 8

PSK
04/29/00

Absolute stability — via Lyapunov Theory

This topic originates with the work of Alexander Lur'e, (1901-?), Russian mathematician from Leningrad/St. Petersburg. The question is easy to state.

Given a linear system with a memoryless nonlinear element in the feedback loop, about which we know very little (say that it lies in a sector), when can we say that the origin is an asymptotically ~~stable equilibrium~~ stable equilibrium for the closed loop system?

[A. I. Luré (1951): Einige nichtlineare Problem aus der Theorie der automatischen Regelung Moscow 1951 (R), transl. Berlin (1957)]

Formally,

Given,
$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -\psi(t, y)$$

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$
$y \in \mathbb{R}^m$

and ψ satisfies, for each $t \geq 0$,

(Sector condition)
$$(\psi(t,y) - K_{min}\, y)^T (\psi(t,y) - K_{max}\, y) \leq 0$$

for K_{min}, K_{max} s.t., $K = K_{max} - K_{min}$ is symmetric and positive definite.

Under what conditions on $G(s) = C(s\mathbb{1} - A)^{-1}B$, K_{min} and K_{max}, can we conclude that the origin is an asymptotically stable equilibrium for the closed loop system.

We will treat this problem in a 2 step process. First we restrict to A Hurwitz and $K_{min} = 0$. Then we consider the original question.

A bit of terminology — <u>sector condition</u> — can be explained by the following lemma

<u>Lemma</u> $\qquad \alpha y^2 \le y\,\psi(y) \le \beta y^2 \qquad\qquad \alpha \le \beta$

is equivalent to

$$(\psi(y) - \alpha y)\,(\psi(y) - \beta y) \le 0$$

<u>Proof</u> $(\Longrightarrow)$ suppose $\alpha y^2 \le y\,\psi(y) \le \beta y^2$.

Then $\quad y\,(\psi(y) - \beta y) \le 0 \quad$ and $\quad y(\psi(y) - \alpha y) \ge 0$

Multiplying these two inequalities,

$$y^2\,(\psi(y) - \beta y)(\psi(y) - \alpha y) \le 0 .$$

But $y^2 \ge 0$.

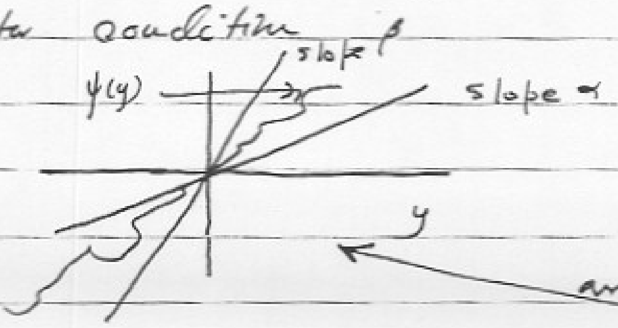
$\quad$ Hence $\;(\psi(y) - \beta y)(\psi(y) - \alpha y) \le 0$

$(\Longleftarrow)$ Multiply the last inequality by y^2 and reverse/retrace the steps $\quad \blacksquare$

<u>Remark</u>. The scalar condition

$$(\psi(y) - \alpha y)\,(\psi(y) - \beta y) \le 0$$

is thus seen to be just the graphical sector condition



Then the condition

$$(\psi(y) - K_{min}\,y)^T(\psi(y) - K_{max}\,y) \le 0$$

is just a multivariable analog of the picture.

<u>Definition</u> Consider the system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -\psi(t,y)$$

where $\forall\, t \geq 0$, $\forall\, y \in \Gamma \subseteq \mathbb{R}^p$, $\mathring{\Gamma}$ connected, $\ni 0$,

$$(\psi(t,y) - K_{min}\,y)^T (\psi(t,y) - K_{max}\,y) \leq 0$$

and $K = K_{max} - K_{min} = K^T > 0$.

The system is <u>absolutely stable</u> with a finite domain Γ ~~if the~~ for the closed loop system, the origin is uniformly, asymptotically, stable with ~~a finite domain~~ any ψ satisfying the sector condit

If $\Gamma = \mathbb{R}^p$, absolute stability $\Leftrightarrow$ global uniform asymptotic stability.

$\mathring{\Gamma}$ = interior of Γ

~~The~~ The main result is the (multivariable) <u>circle criterion</u>. The idea of the proof is to show that under suitable hypotheses one has a <u>time-independent</u> quadratic Lyapunov function. The key ideas here have to do with the concept of <u>passivity</u>.

Recall that mechanical systems <u>without</u> friction can be cast in the hamiltonian form

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial x} + f$$

Here f is an external (generalized) force corresponding to the (generalized) coordinate x. Now we *define* the rate at which mechanical work is done by the external force applied to the system as,

$$\langle f, \dot{x} \rangle .$$

Then

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p}$$

$$= \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \left(-\frac{\partial H}{\partial x} + f \right)$$

$$= \frac{\partial H}{\partial p} \cdot f$$

$$= \dot{x} \cdot f$$

$$= \text{rate of work } (= \text{power}).$$

So energy stored in the system (ie H) increases at a rate = power input.

If there is internal dissipation then

$$\frac{dH}{dt} \leq \text{power},$$

the *dissipation inequality*.

A *passive system* is one that satisfies the dissipation inequality. Treating forces as inputs and (generalized) velocities as outputs, we write the dissipation inequality as

$$H(x(t), p(t)) \le H(x(0), p(0)) + \int_0^t y^T(\sigma) f(\sigma)\, d\sigma$$

Definition A system is passive if

$$\int_0^{\tilde{T}} y^T(\sigma) \underbrace{f(\sigma)}_{\text{input}}\, d\sigma \ge 0 \qquad \forall\ \tilde{T} \ge 0.$$

This definition is an abstract one for the general setting of input–output systems. For hamiltonians that are a priori bounded below (say $H(x, p) \ge c$) we see that

$$\begin{aligned} 0 &\le H(x(t), p(t)) - c \\ &\le H(x(0), p(0)) - c + \int_0^t y^T(\sigma) f(\sigma)\, d\sigma \\ &\le \delta + \int_0^t y^T(\sigma) f(\sigma)\, d\sigma \end{aligned}$$

which says

$$\int_0^t y(\sigma)^T f(\sigma) \quad \text{is bounded below}$$

(by a constant $-\delta$ that depends on initial conditions), $\forall\ t \ge 0$.

Definition A $p \times p$ matrix $Z(s)$ of transfer functions is <u>positive real</u> if

$Z(s)$ has all matrix elements analytic in $\{s;\ Re(s) \geqslant 0\}$

$Z^*(s) = Z(s^*)$ for $\{s:\ Re(s) > 0\}$ and

$Z^T(s^*) + Z(s)$ is positive semidefinite for $\{s:\ Re(s) > 0\}$, where $(*)$ denotes complex conjugation and superscript T denotes matrix transpose.

$Z(s)$ is strictly positive real if $Z(s-\varepsilon)$ is positive real for some $\varepsilon > 0$.

Remark Positive real transfer functions are impedance or admittance matrices made of linear resistors, capacitors, inductors, transformers and gyrators.

Lemma (Kalman - Yacubovitch - Popov)

Let $Z(s) = C(s\mathbb{1} - A)^{-1}B + D$ be the $p \times p$ transfer function of the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where A is Hurwitz, (A, B) is controllable

PSK (04/30/00)

(A, C) is observable. Then Z is strict positive real iff there exists $P = P^T > 0$ and matrices W, L and constant $\varepsilon > 0$ such that

$$A^T P + PA = -L^T L - \varepsilon P$$

$$PB = C^T - L^T W$$

$$W^T W = D + D^T$$

Proof (Sufficiency)

Suppose there exist P, L, W, ε satisfying the above equations. Take $\mu \in (0, \frac{\varepsilon}{2})$.

$$(A + \mu \mathbb{1})^T P + P(A + \mu \mathbb{1}) = -L^T L - (\varepsilon - 2\mu) P$$

$P > 0$ and $L^T L + (\varepsilon - 2\mu) P > 0$. Then by standard matrix Lyapunov theory (e.g. see Theorem 5.36 Lyapunov lemma on page 211 of Sastry), the matrix $(A + \mu \mathbb{1})$ is Hurwitz. Hence $Z(s-\mu)$ is analytic in $\{s : Re(s) \geq 0\}$.
Let $\Phi(s) = (s\mathbb{1} - A)^{-1}$

$$Z(s-\mu) + Z^T(-s-\mu) = D + D^T + C\Phi(s-\mu)B + B^T \Phi^T(-s-\mu) C^T$$

substituting $C = (PB + L^T W)^T$ and $D + D^T = W^T W$ we get,

$$Z(s-\mu) + Z^T(-s-\mu) = W^T W + (B^T P + W^T L)\Phi(s-\mu) B + B^T \Phi^T(-s-\mu)(PB + L^T W)$$

$$
\begin{aligned}
= \; & W^T W + W^T L\,\Phi(s-\mu)B + B^T\Phi^T(-s-\mu)L^T W \\
& + B^T\Phi^T(-s-\mu)PB \\
& + B^T P\,\Phi(s-\mu)B
\end{aligned}
$$

← Fill in the rest →

PSK (04/30)

Theorem < Multivariable Circle Criterion (Hurwitz case) >

$A : \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R}^m \to \mathbb{R}^n$, $C : \mathbb{R}^n \to \mathbb{R}^m$

Let $[A, B, C]$ be a controllable and observable triple. Let A be Hurwitz. Suppose ψ satisfies the sector condition

$$\psi^T(t,y)\,(\psi(t,y) - Ky) \leq 0 \qquad \forall\, t \geq 0,\ y \in \mathbb{R}^m$$

$$K = K^T > 0$$

Then the closed loop system

$$(*) \quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\psi(t,y) \end{cases}$$

is absolutely stable <u>provided</u>

$$Z(s) = \mathbb{1}_m + K\,G(s)$$

is strict positive real.

If the sector condition is only valid for $T \subset \mathbb{R}^r$, $0 \in \mathring{T}$, then the strict positive reality of $Z(s)$ ensures only that the closed loop system is absolutely stable with finite domain.

$\mathring{T}$ = interior of T

<u>Proof</u> $Z(s) = \mathbb{1}_m + K\,G(s)$ is the transfer function of the linear system

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

$$\tilde{y} = KC\tilde{x} + \tilde{u}.$$

Setting $D = D^T = \mathbb{1}_m$ in the Kalman Yacubovitch Popov lemma & also further replacing C in the lemma by KC one concludes that there exists $P = P^T > 0$ and

and $\varepsilon > 0$

matrices L and W such that

$$A^T P + PA = -L^T L - \varepsilon P$$

$$PB = (KC)^T - L^T W$$

$$W^T W = D + D^T = 2 \mathbb{1}_m$$

Take $W = \sqrt{2}\, \mathbb{1}_m \Rightarrow PB = C^T K - \sqrt{2} L^T$.

Now consider the function

$$V(x) = x^T P x .$$

Along trajectories of the closed loop system $(*)$,

$$\dot V = \dot x^T P x + x^T P \dot x$$

$$= (Ax - B\psi(t, Cx))^T P x + x^T P (Ax - B\psi(t,Cx))$$

$$= x^T (A^T P + PA) x - 2 x^T PB \psi(t, Cx) .$$

Since $-2\psi^T(t,y)(\psi(t,y) - Ky) \geq 0$ by the sector condition, it follows that

$$\dot V \leq x^T(A^T P + PA)x - 2x^T PB\psi - 2\psi^T(\psi - KCx)$$

$$= x^T (A^T P + PA) + 2x^T (C^T K - PB)\psi - 2\psi^T\psi$$

(by KYP) $$= -\varepsilon x^T P x - x^T L^T L x + 2\sqrt{2}\, x^T L^T \psi - 2\psi^T \psi$$

$$= -\varepsilon x^T P x - (Lx - \sqrt{2}\psi)^T (Lx - \sqrt{2}\psi)$$

$$\leq -\varepsilon x^T P x$$

The function V satisfies all the hypotheses of the Time Varying Lyapunov Theorem (see Lecture Notes Lecture 6, part(i), page 7) with $\alpha_1(r) = \lambda_{min}(P) r^2$ $\alpha_2(r) = \lambda_{max}(P) r^2$ and $\alpha_3(r) = \varepsilon \lambda_{min}(P) r^2$.

So the closed loop system has the origin as an uniformly asymptotically stable (in fact exponentially stable) equilibrium point. $\boxtimes$

$\alpha_i = c_i r^2$
$i = 1,2,3$

Remark: We have used the easily shown fact that $[A, C]$ observable $\Leftrightarrow$ $[A, KC]$ is observable for K any nonsingular matrix

Remark Suppose A is not Hurwitz. It is possible that there exists a K_{min} such that the matrix $(A - BK_{min}C)$ is Hurwitz (under the assumption that $[A, B]$ is controllable and $[A, C]$ is observable). (In fact, conditions for this are difficult to determine and there is a deep problem hidden here — work of Byrnes, Brockett, Rosenthal and others. We will sweep these difficulties under the rug!)

This work involves the methods of algebraic geometry including the Schubert calculus

Now the closed loop system of

$$\dot{x} = Ax + Bu \; ; \quad y = Cx \; ; \quad u = -\psi(t, y) \quad \text{is}$$

$$\begin{aligned} \dot{x} &= Ax - B\psi(t, Cx) \\ &= (A - BK_{min}C)x - B(\psi(t, Cx) - K_{min}Cx) \end{aligned}$$

which is the closed loop system of

$$\dot{x} = (A - BK_{min}C)x + Bu \; ; \quad y = Cx \; ; \quad u = -\tilde{\psi}(t, y)$$

where,

$$\tilde{\psi}(t, y) \triangleq \psi(t, y) - K_{min}y \, .$$

Note that the triple $[A, B, C]$ is controllable and observable iff the triple $[A - BK_{min}C, B, C]$ is controllable and observable. (exercise in linear algebra)