2011 spring

ENEE 661 Lecture 1(a)

Nonlinearity

In this course we will discuss nonlinear control theory from the point of view of understanding the main principles and techniques that shed light on qualitative properties of such systems. We will address:

(i) controllability (when does there exist a control that drives the system from an initial state to a prescribed target state?)

(ii) observability (can you infer state from observations of an output signal?)

(iii) special solutions (equilibria, periodic orbits) and bifurcations with respect to parameter variation.

(iv) stability of solutions (a central topic)

Further we will discuss how this understanding leads to approaches for design

Our techniques will include algebraic, geometric and analytic methods in the study of differential equations.

Nonlinearity arises in a number of ways:

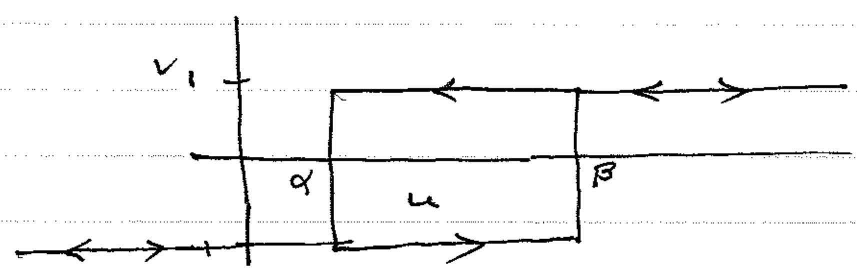
(i) State space is not a vector space. For instance, in the control of a magnetic moment using external fields, the state space is a sphere $\{ (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = r^2 \}$

(ii) Equations of motion are nonlinear. For instance the pendulum

$$\ddot{\theta} + \frac{g}{l} \sin(\theta) = u$$

where u = torque applied at pivot.

(iii) actuators (or sensors) are subject to nonlinear constitutive relations - e.g. hysteresis

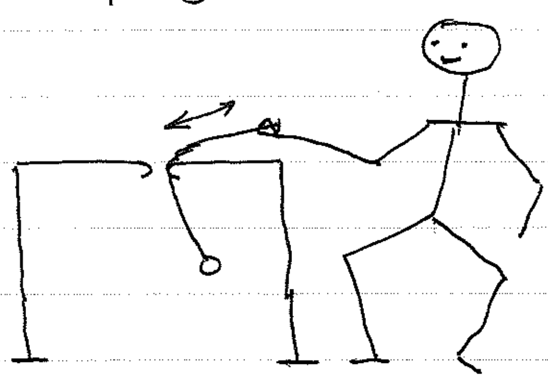


<u>Increasing</u> u from $-\infty$ to β leaves v constant $= -1$ until a jump occurs for $u = \beta$ and thereafter v remains at 1 for further <u>increase</u> in u.

<u>Decreasing</u> u from $+\infty$ to α leaves v = constant $= +1$ until a jump occurs at $u = \alpha$ and thereafter v remains at -1 for further <u>decrease</u> in u

Magnetic recording processes depend on hysteresis. Other applications of hysteresis arise actuators incorporating deformable materials.

<u>Example 1</u> Consider the controlled pendulum in the adjoing figure.



The pendulum is suspended on a string fed through a hole on a table top and controlled by an investigator.

The investigator controls the length of the pendulum (possibly periodically). The interaction of the pendulum with the table introduces a <u>frictional torque</u>. Approximating $\sin(\theta)$ by θ (small oscillation assumption), we obtain the model (with damping constant $b > 0$)

$$\ddot{x} + v\,x = -b\dot{x}$$

where $v = \frac{g}{l}$ is interpreted as a control that depends on the time function used by the investigator. We thus have a state space model

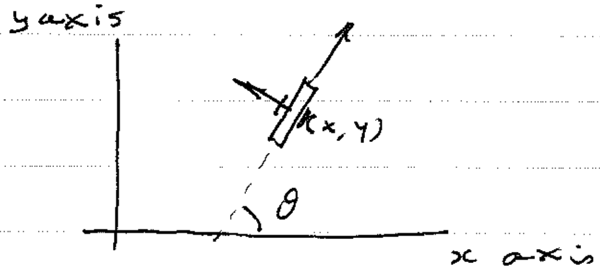
$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + v(t)\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\Longleftrightarrow \quad \dot{z} = Az + vBz$$

where $A = \begin{pmatrix} 0 & 1 \\ 0 & -b \end{pmatrix}$; $B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$.

Here the control enters multiplicatively. If $v = \text{constant}$ then dynamics describes free, damped oscillation of a pendulum with natural frequency $= \sqrt{v}$.

Example 2 Consider the unicycle seen from above in the adjoining figure



Forward speed (by pedaling) is u. Steering rate is ω. It is then easy to show that

$$\dot{x} = u \cos(\theta)$$
$$\dot{y} = u \sin(\theta)$$
$$\dot{\theta} = \omega.$$

We can repackage this as

$$\dot{g} = g\,\xi$$

where

$$g = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\xi = \omega \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + u \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Matrices of the form g above constitute a matrix (Lie) group with the multiplication

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & x \\ \sin(\theta) & \cos(\theta) & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\phi) & -\sin\phi & x' \\ \sin(\phi) & \cos\phi & y' \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & x + x'\cos\theta - y'\sin\theta \\ \sin(\theta+\phi) & \cos(\theta+\phi) & y + x'\sin\theta + y'\cos\theta \\ 0 & 0 & 1 \end{pmatrix}$$

and inverse

$$g^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) & -x\cos\theta - y\sin\theta \\ \sin(-\theta) & \cos(-\theta) & x\sin\theta - y\cos\theta \\ 0 & 0 & 1 \end{pmatrix}.$$

The collection of all such g constitutes the rigid motion group in the plane $SE(2)$. Formally,

$$SE(n) = \left\{ \left[\begin{array}{c|c} A & b \\ \hline \underline{0} & 1 \end{array}\right] : \text{ where } A^T A = I_n \right\}$$

$b \in \mathbb{R}^n$, $\det(A) = 1$ and $\underline{0}$ = row of n zeros

The block $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $n=2$ is just a planar counterclockwise rotation by θ.

Thus motion of unicycle on the plane gives a curve in $SE(2)$ with two controls ω and u. If the ~~curves~~ controls are set $= 0$ then there is no motion, i.e. we have a drift-free system.

$SE(2)$ is <u>not</u> a vector space. It is an example of a smooth manifold.