Lecture 2 (i) (matrix Lie groups and Lie algebras – introduction)

Definition 1 Recall that a set S together with a multiplication operation denoted by '$\cdot$' $\cdot : S \times S \to S$, is a group if the following axioms hold:

(i) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall\, a, b, c \in S$

(ii) there is an element $e \in S$ such that $a = e \cdot a = a \cdot e \quad \forall\, a \in S$
(e is called the identity ; an identity if it exists, is unique)

(iii) for each $a \in S$ there is an element b such that $a \cdot b = b \cdot a = e$
It can be shown that a such a 'b' is uniquely determined by 'a' and we denote 'b' as a^{-1}.

We call the pair $G = (S, \cdot)$ a group.

Example 2. $G = (Gl(n, \mathbb{R}), \cdot)$ where $Gl(n, \mathbb{R})$ denotes the set of all $n \times n$ nonsingular matrices with matrix multiplication defining the group structure. This is the general linear group.

Definition 3. A subset $Q \subset S$ where $G = (S, \cdot)$ is a group can also inherit the group structure from G, provided,

(i) $a, b \in Q \Rightarrow a \cdot b \in Q$

(ii) e the identity element in S actually is in Q

(iii) $a \in Q \Rightarrow a^{-1} \in Q$.

In that case we call $\tilde{G} = (Q, \cdot)$ a <u>subgroup</u> of $G = (S, \cdot)$

Example 4. $O(n, \mathbb{R})$ the set all $n \times n$ real orthogonal matrices is a subgroup of $Gl(n, \mathbb{R})$.

Example 5. Let $SO(n, \mathbb{R}) = \{ M \in O(n, \mathbb{R}) \mid \det(M) = 1 \}$

Then $SO(n, \mathbb{R})$ is a subgroup of $O(n, \mathbb{R})$. It is the <u>special</u> <u>orthogonal</u> <u>group</u>.

Definition 6 A group G is <u>abelian</u> if $a \cdot b = b \cdot a$ $\forall\ a, b \in G$

Example 7 $G = (\mathbb{R}, +)$, $G = (\mathbb{R}^n, +)$, $G = (Mat(n), +)$ $G = SO(2, \mathbb{R})$, are all <u>abelian</u> groups. $Gl(n, \mathbb{R})$ for $n \geq 2$ is not abelian.

Definition 8 Given two groups $G_1 = (S_1, \cdot_1)$ and $G_2 = (S_2, \cdot_2)$ we define the <u>direct product</u> of these two groups to be

$$G = (S, \cdot)$$

where $S = S_1 \times S_2$ (Cartesian product of sets

and $(a_1, a_2) \cdot (b_1, b_2) = (a_1 \cdot_1 b_1, a_2 \cdot_2 b_2)$

Direct products gives us a way to define new groups out of <u>building blocks</u>.

Example 9 Lets $G_1 = (SO(2), \cdot)$ and $G_2 = (\mathbb{R}^2, +)$.

Then,

$$G = SO(2) \times \mathbb{R}^2$$

with multiplication

$$\left(\begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}, \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix} \right)$$

Contrast this with the group $SE(2, \mathbb{R})$, encountered in the discussion of the <u>unicycle</u> in Lecture 1(a) (page 5). The groups are NOT the same since the multiplication rules are different. $G = SO(2) \times \mathbb{R}^2$ derives its multiplication rule from combining the multipli-cation in $SO(2)$ and the <u>vector addition</u> in $\mathbb{R}^2$. In contrast the <u>semi-direct product</u> $SE(2, \mathbb{R})$ derives its multiplication rule as a subgroup of $Gl(3, \mathbb{R})$.

The matrix groups encountered so far are all <u>subgroups</u> of $Gl(n, \mathbb{R})$ which in turn is an <u>open subset</u> of $Mat(n, \mathbb{R})$ the set of all $n \times n$ matrices over the reals.

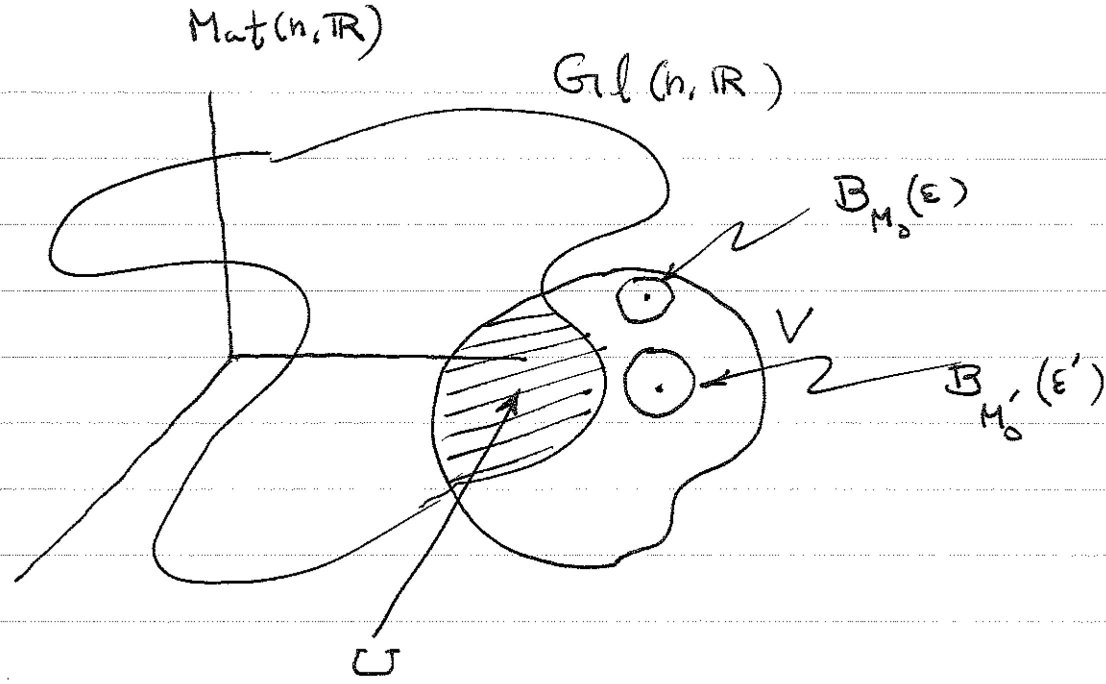
because of the condition $\det(X) \neq 0$

$Mat(n, \mathbb{R})$ is clearly a vector space of dimension n^2 and can be equipped with metrics (from norms) in a number of different ways. For instance a ball $B_{M_0}(\varepsilon)$ of radius $\varepsilon > 0$ centered at M_0 in $Mat(n, \mathbb{R})$ can be defined to be

$$B_{M_0}(\varepsilon) = \left\{ M \in Mat(n, \mathbb{R}) : \left(tr\left((M - M_0)^T (M - M_0)\right)\right)^{1/2} < \varepsilon \right\}$$

This is the open Euclidean ball in $Mat(n, \mathbb{R})$ defining what is known as the <u>usual topology</u>. $Gl(n, \mathbb{R})$ <u>inherits this topology</u> by the definition:

$U \subset Gl(n, \mathbb{R})$, is an <u>open set</u> in $Gl(n, \mathbb{R})$ iff $U = Gl(n, \mathbb{R}) \cap V$, where V is an open subset of $Mat(n, \mathbb{R})$; <u>and</u> V is an open subset of $Mat(n, \mathbb{R})$ iff for each $M_0 \in V$, there is an $\varepsilon = \varepsilon(M_0) > 0$ such that $B_{M_0}(\varepsilon) \subset V$ is a strict subset of V. The adjoining figure should help.



Observe that the definition of $SO(n,\mathbb{R})$ as a subgroup of $Gl(n,\mathbb{R})$ allows us to similarly introduce the <u>subspace topology</u> on $SO(n;\mathbb{R})$:

$V \subset SO(n;\mathbb{R})$ is open iff

$V = SO(n;\mathbb{R}) \cap U$

where $U \subset Gl(n;\mathbb{R})$ is open.

All subgroups of $Gl(n;\mathbb{R})$ inherit a topology in this way.

One can actually show more: $Gl(n;\mathbb{R})$ can be given the structure of a manifold i.e. open sets can be used to cover $Gl(n;\mathbb{R})$ in such a way as to yield coordinate charts i.e. one has $\{(U_\alpha, \varphi_\alpha): U_\alpha \subset Gl(n,\mathbb{R})$ open and $\varphi_\alpha : U_\alpha \to \mathbb{R}^k$ smooth map, invertible and with smooth inverse, $\alpha \in A =$ index set$\}$. This is a starting point for thinking about $Gl(n;\mathbb{R})$ as a smooth manifold. We postpone details for later.

Of great importance to physical problems are the classical groups (and their subgroups). We list them over the reals $\mathbb{R}$ and the complexes $\mathbb{C}$.

Over $\mathbb{R}$

general linear group $Gl(n,\mathbb{R}) = \{X : X \ n\times n$ matrix, and $\det(X) \neq 0\}$

special linear group $Sl(n,\mathbb{R}) = \{X : X \in Gl(n,\mathbb{R})$ and $\det(X) = 1\}$

Orthogonal group $O(n,\mathbb{R}) = \{X : X \in Gl(n,\mathbb{R}),\ X^T X = \mathbb{1}_n$ the identity $\}$

Special orthogonal group $SO(n,\mathbb{R}) = O(n,\mathbb{R}) \cap Sl(n,\mathbb{R})$

Symplectic group $Sp(2n,\mathbb{R})$

$$= \{X : X \in Gl(2n,\mathbb{R}),\quad X^T J X = J\}$$

Here $J = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$

Pseudo orthogonal group $O(p,q,\mathbb{R})$

$$= \{X : X \in Gl(p+q,\mathbb{R}),\ X^T \Sigma_{p,q} X = \Sigma_{p,q}\}$$

Here $\Sigma_{p,q} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix}$.

Over $\mathbb{C}$, one replaces the transpose operation by the Hermitian transpose * or complex conjugate transpose. Thus, in particular, we refer to

$U(n, \mathbb{C}) = \{ X : X \in Gl(n, \mathbb{C}), \ X^* X = \mathbb{1}_n \}$
is the <u>unitary group</u>, and
$SU(n, \mathbb{C}) = Sl(n, \mathbb{C}) \cap U(n, \mathbb{C})$
is the <u>special unitary group</u>.

Since all the groups above are imbedded in the space $Mat(n, \mathbb{R})$ (or $Mat(n, \mathbb{C})$), it makes sense to speak of <u>curve</u> in a classical group that is continuously differentiable with respect to its parameter. Thus, consider

$$t \longmapsto \Phi(t) \in SO(n, \mathbb{R})$$

a differentible curve for $t \in [0, T]$.

Then $\Phi^T(t) \Phi(t) \equiv \mathbb{1}_n \quad \forall t \in [0,T]$

Differentiating both sides, we get,

$$\dot{\Phi}^T(t) \, \Phi(t) + \Phi^T(t) \, \dot{\Phi}(t) \equiv 0$$

$$\Rightarrow \quad \left(\Phi^T(t) \, \dot{\Phi}(t) \right)^T + \left(\Phi^T(t) \, \dot{\Phi}(t) \right) \equiv 0$$

$\Rightarrow \quad \Phi^T(t) \, \dot{\Phi}(t) = \xi(t)$ $n \times n$ skew symmetric matrix valued function of t.

Equivalently, $\dot{\Phi}(t) = \Phi(t) \, \xi(t)$,
since $\left(\Phi^T(t) \right)^{-1} = \left(\Phi^{-1}(t) \right)^{-1} = \Phi(t)$.

Thus, to each smooth curve in $SO(n)$, one can associate a smooth curve in $so(n)$, the space of $n \times n$ skew-symmetric matrices. Conversely, given any continuous curve $\xi(t)$ in $so(n)$ and $\Phi(0) \in SO(n)$, one can produce (by integration) an unique curve $\Phi(t)$ in $SO(n)$. The proof of this converse is not so obvious. But we can see it easily in a special case $\xi(t) \equiv \xi$ a constant skew symmetric matrix. In that case, by the theory of linear differential equations,

$$\Phi(t) = \Phi(0)\, e^{t\xi}.$$

Hence

$$\begin{aligned} \Phi(t)\,\Phi^T(t) &= \Phi(0)\, e^{t\xi}\, e^{t\xi^T}\, \Phi^T(0) \\ &= \Phi(0)\, e^{t\xi}\, e^{-t\xi}\, \Phi^T(0) \\ &= \Phi(0)\, e^{t(\xi-\xi)}\, \Phi^T(0) \\ &= \mathbb{1}_n, \quad \forall\, t \in [0,T]. \end{aligned}$$

To prove the converse in general for time dependent ξ, one needs a representation of the solution to the differential equation

$$\dot{\Phi}(t) = \Phi(t)\, \xi(t).$$

See Wei-Norman (1964) paper.

In a similar vein, consider $t \mapsto \Phi(t)$ a smooth curve in $Sp(2n, \mathbb{R})$. Then

$$\Phi^T(t) \, J \, \Phi(t) \equiv J$$

Differentiating both sides, we get

$$\dot{\Phi}^T J \Phi + \Phi^T J \dot{\Phi} \equiv 0$$

$$\leftrightarrow \quad -\dot{\Phi}^T J^T \Phi + \Phi^T J \dot{\Phi} \equiv 0$$

(since $-J^T = J$)

$$\leftrightarrow \quad -\left(\Phi^T J \dot{\Phi}\right)^T + \Phi^T J \dot{\Phi} \equiv 0$$

Thus $\Phi^T J \dot{\Phi} = \tilde{\xi}(t)$ a <u>symmetric</u> matrix-valued function. Note that

$$(J \tilde{\xi})^T J + J (J \tilde{\xi})$$

$$= \tilde{\xi}^T J^T J + J J \tilde{\xi}$$

$$= \tilde{\xi} - \tilde{\xi}$$

(since $\tilde{\xi} = \tilde{\xi}^T$ and $J^T J = \mathbb{1}_{2n}$, $J J = -\mathbb{1}_{2n}$)

$$= 0$$

Hence $J \tilde{\xi}(\cdot) : [0, T] \to sp(2n)$
where $sp(2n) = \{ X : X^T J + J X = 0 \}$

We call $sp(2n)$ the space of <u>hamiltonian</u> (or infinitesimally symplectic) matrices. It is clearly a vector space, and since

$$\Phi^T J \dot{\Phi} = \tilde{\xi}(t)$$

$$\Leftrightarrow \quad \dot{\Phi} = J^{-1}(\Phi^T)^{-1}\tilde{\xi}$$

$$= -\Phi J \tilde{\xi} \triangleq \Phi \xi$$

$\xi = -J\tilde{\xi} \in sp(2n)$

(since $\Phi^T J \Phi = J$ and $J^{-1} = -J$, $J^T = -J$),

it follows that $sp(2n)$ plays the same role for $Sp(2n)$, as does $so(n)$ for $SO(n)$. In particular, if $\xi(t) \equiv \xi$ constant $\in sp(2n)$ then,

$$t \mapsto \exp(t\xi) \in Sp(2n)$$

$\forall\, t \in \mathbb{R}$. The above construction is applicable to all the classical groups.

<u>Definition 10</u>

$gl(n,\mathbb{R})$ = all $n \times n$ matrices

$sl(n,\mathbb{R}) = \{X : X \in gl(n,\mathbb{R}),\ tr(X) = 0\}$

$so(n,\mathbb{R}) = \{X : X \in gl(n,\mathbb{R}),\ X^T + X = 0\}$

$sp(2n,\mathbb{R}) = \{X : X \in gl(2n,\mathbb{R}),\ X^T J + J X = 0\}$

$so(p,q,\mathbb{R}) = \{X : X \in gl(p+q,\mathbb{R}),\ X^T \Sigma_{p,q} + \Sigma_{p,q} X = 0\}$

These vector spaces have the important property that

$$
\begin{aligned}
X \in gl(n) &\Rightarrow \exp(X) \in Gl(n) \\
X \in sl(n) &\Rightarrow \exp(X) \in Sl(n) \\
X \in so(n) &\Rightarrow \exp(X) \in SO(n) \\
X \in sp(2n) &\Rightarrow \exp(X) \in Sp(2n) \\
X \in so(p,q) &\Rightarrow \exp(X) \in SO(p,q)
\end{aligned}
$$

The exponential map takes value in appropriate classical groups. But, in general, it is <u>not onto</u>. For example, there <u>does not exist</u> a <u>real</u> matrix X such that

$$\exp(X) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \in Gl(2, \mathbb{R})$$

Since, $\frac{d}{dt} \exp(tX) = X \exp(tX)$

$= X$ for $t = 0$

and $\exp(0 \cdot X) = \mathbb{1}$, we interpret $gl(n)$, $sl(n)$, $so(n)$, $sp(2n)$, $so(p,q)$, as the spaces where <u>velocities of curves</u> passing through identity in corresponding classical groups live. <u>These vector spaces also carry another, algebraic structure</u>

<u>Definition 11</u>

A vector space V, together with an operation (Lie bracket)

$$[\cdot,\cdot] : V \times V \to V$$

$$(a, b) \mapsto [a, b]$$

is said to constitute a Lie algebra $\mathfrak{g} = (V, [\cdot,\cdot])$ if the operation above satisfies the axioms

(i) $[a, b] = -[b, a]$

(ii) $[\lambda a + \mu b, c] = \lambda [a, c] + \mu [b, c]$

where $\lambda, \mu \in F$ the underlying field of scalars for V, and

(iii) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

Jacobi identity.

Defining $[X, Y] = XY - YX$ the matrix commutation for matrices X, Y, each of the spaces $gl(n)$, $sl(n)$, $so(n)$, $sp(2n)$, $so(p,q)$ is a <u>Lie algebra</u>

These are the classical Lie algebras.

Definition 12

For any subgroup $G \subseteq Gl(n)$, we define the associated Lie algebra to be the vector space

$$\mathfrak{g} = \{ X \in gl(n) : \exp(tX) \in G \ \forall t \in \mathbb{R} \}$$

See Theorem 17 in R. Howe, <u>Very Basic Lie Theory</u>.

Definition 13. Given a set of matrices $\{A_1, A_2, \dots, A_k\}$ of size $n \times n$, we define

$$\mathfrak{g} = L.A. \{ A_1, A_2, \dots, A_k \}$$

to be the smallest Lie algebra generated by $A_1, A_2, \dots, A_k$ if

(i) the underlying vector space contains the linear span of $\{A_1, A_2, \dots, A_k\}$

(ii) is closed under Lie bracket

(iii) there is no lower dimensional ~~sub~~space satisfying (i) and (ii).

The dimension of a Lie algebra is the dimension of the underlying vector space. A Lie algebra of $n \times n$ matrices, being necessarily a subspace of $gl(n)$, has dimension at most $= n^2$.

Let $\mathfrak{g} = (V, [\cdot, \cdot])$ and let $\xi_1, \xi_2, \ldots, \xi_m$ constitute a basis for V. Then,

$[\xi_i, \xi_j]$ being an element of V, can be <u>uniquely</u> written as a linear combination of $\xi_1, \xi_2, \cdots, \xi_m$,

$$[\xi_i, \xi_j] = \sum_{k=1}^{m} T_{ij}^{k} \xi_k$$

The numbers T_{ij}^{k} are called structure constants of the Lie algebra in that basis.

Exercise 14 What are the dimensions of $sl(n)$, $so(n)$, $sp(2n)$, $so(p,q)$?