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# ENEE 661

# Nonlinear Control Systems

## Lecture 1.

The primary goals of this course are:

(a) to provide an exposition of the basic features of nonlinear dynamics;

(b) formulate stability concepts, and techniques to assess stability;

(c) develop the connections between feedback and stability;

(d) and organize analysis and design methods leading to feedback design, constructive controls and applications in physical systems.

Some of the salient aspects of nonlinearity include:

- nonintegrability (difficult calculus problems)
- multiple (isolated) equilibria
- limit cycles
- finite escape time possibility
- subharmonics
- chaos and nonintegrability (complicated dynamics)
- multiple modes of behavior (bifurcations)

(1) The linear constant coefficient system

$$\frac{dx}{dt} = A\, x(t)$$

where $A$ is an $n \times n$ matrix and $x(t) \in \mathbb{R}^n$, has the solution

$$x(t) = e^{At} x_0$$

corresponding to the initial condition

$$x(0) = x_0 .$$

The elements of the matrix $e^{tA}$ are completely expressible in terms of the elementary functions – exponentials, polynomials, and sines and cosines –, of time $t$. In contrast, the equation of a pendulum

$$\ddot{\theta} + \omega^2 \sin(\theta) = 0$$

(where $\omega = \sqrt{\frac{g}{\ell}}$, $g$ = acceleration due to gravity, $\ell$ = length of the pendulum)

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of the form,

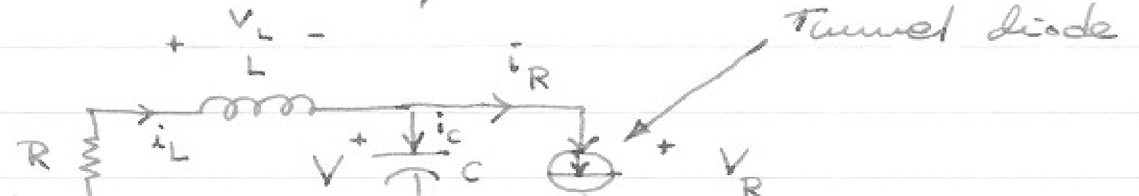
$$\int \frac{d\theta}{\sqrt{2(E - \omega^2(1-\cos\theta))}}$$

to solve the problem.

(2) Generically the linear system of the example 1 above has a single equilibrium, $x = 0$, since, if we were to pick a matrix $A$ at random, it is likely to be nonsingular. If it happens to be singular, the problem has a continuum $Ker(A)$ of (non-isolated) equilibria.

In contrast when interpreting the phrase 'generic' correctly, nonlinear systems have multiple isolated equilibria. We illustrate this.

Consider the circuit with a tunnel diode shown in Figure 1 below.



The tunnel diode characteristic is given by $i_R = h(V_R)$, see Figure 2.
For the linear components in the circuit,

$$V_L = L \frac{d i_L}{dt}$$

$$i_C = C \frac{d V_C}{dt}$$

and we have the Kirchhoff laws

(KVL) $$E = V_L + V_C + i_L R$$

(KCL) $$i_R = i_L - i_C$$

Noting that $V_C = V_R$, and denoting $x_1 \triangleq V_C$, $x_2 \triangleq i_L$, we have the equations

$$\dot{x}_1 = \frac{1}{C}\left( x_2 - h(x_1) \right)$$

$$\dot{x}_2 = \frac{1}{L}\left( E - x_1 - x_2 R \right).$$

The equilibria of this state-space model are precisely the intersection points

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$x_2 = h(x_1)$.

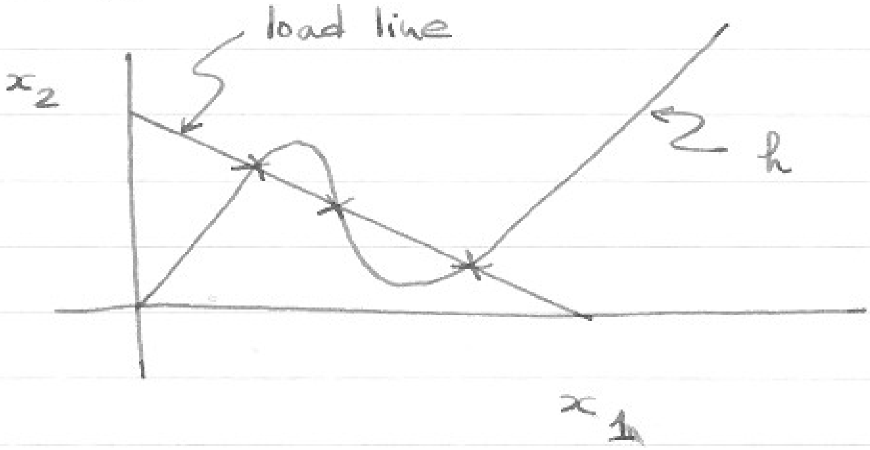



Figure 2.

Clearly there exist multiple, isolated equilibria and they persist in the sense that sufficiently small changes in the diode characteristics or the load-line do not change the number of equilibria. (In this context, the specific values of L and C have no role.) → **but are nonzero**

(3) The linear system,

$$\dot{x}_1 = x_2$$

other solutions are periodic satisfying,

$$x_1^2(t) + x_2^2(t) = x_1^2(0) + x_2^2(0) = \text{constant}.$$

There are no solutions <u>tending to</u> a periodic solution in the limit as $t \to \infty$. In fact, such limit cycle behavior is not possible in linear constant coefficient systems of any dimension. In contrast, nonlinear systems permit limit cycle behavior, as in the example of the negative resistance oscillator ( Figure 3 ). The voltage-controlled characteristic of the negative resistance device is given in Figure 4.

From the Kirchhoff current law, $i_C + i_L + i = 0$, and the linear capacitor and inductor dynamics, one writes down,

$$C\frac{dV}{dt} + \frac{1}{L}\int_{-\infty}^{t} V(s)\,ds + h(V) = 0,$$

or, by differentiating this,

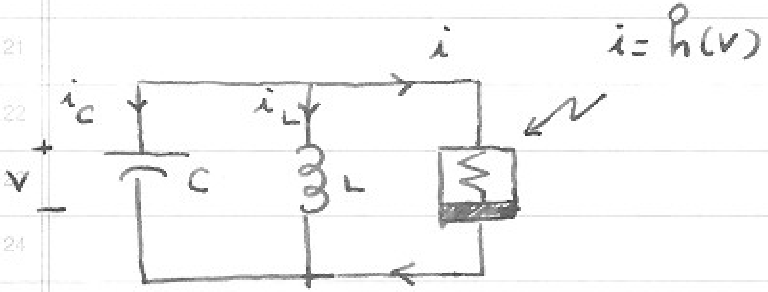
setting $\tau \triangleq \frac{t}{\sqrt{LC}}$, using $\dot{(\;)}$ to denote $\frac{d}{d\tau}$, and letting $\epsilon = \sqrt{\frac{L}{C}}$, we obtain,

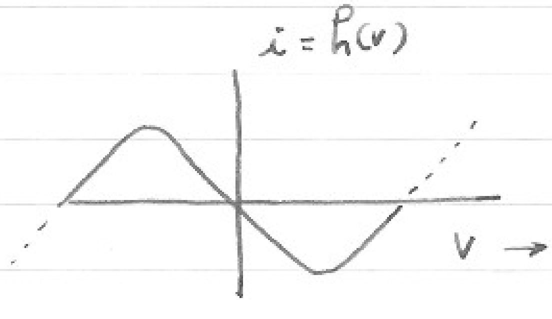
$$\ddot{v} + \epsilon\, h'(v)\,\dot{v} + v = 0.$$

This, in the special case $h(v) = -v + \frac{v^3}{3}$, yields the famous ~~van der Pol~~ van der Pol oscillator

$$\ddot{v} + \epsilon\,(v^2 - 1)\,\dot{v} + v = 0.$$

The limit cycle in this system has been explored exhaustively. You can discover it numerically).



The state model for the van der Pol equation, given by setting $x_1 \triangleq i_L$ and $x_2 \triangleq v$ is

$$\varepsilon \dot{x}_1 = x_2$$
$$\dot{x}_2 = -\varepsilon \left[ x_1 - x_2 + \tfrac{1}{3} x_2^3 \right]$$

The total energy stored in the inductor and capacitor is

$$\mathcal{E} = \frac{1}{2} L i_L^2 + \frac{1}{2} C v^2$$

For a general $h$, $i_L = -i - i_C$

$$= -h(v) - C\frac{dv}{dt} = -h(v) - \frac{C}{\sqrt{LC}}\dot{v}$$
$$= -h(v) - \frac{\dot{v}}{\varepsilon}$$

Thus $\frac{1}{2} L i_L^2 = \frac{1}{2} L \left(h(v) + \frac{\dot{v}}{\varepsilon}\right)^2$

$$= \frac{1}{2} C \left(\varepsilon h(v) + \dot{v}\right)^2$$

Thus, $\mathcal{E} = \frac{1}{2} C \left(\varepsilon h(v) + \dot{v}\right)^2 + \frac{1}{2} C v^2$.

If we use the alternate set of state variables

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -\varepsilon h'(z_1) z_2 - z_1 ,$$

One concludes that

$$\mathcal{E} = \frac{1}{2} C \left[ z_1^2 + (\varepsilon h(z_1) + z_2)^2 \right]$$

and

$$\dot{\mathcal{E}} = \frac{\partial \mathcal{E}}{\partial z_1} \dot{z}_1 + \frac{\partial \mathcal{E}}{\partial z_2} \dot{z}_2$$

$$= -\varepsilon C z_1 h(z_1)$$

The behavior of $\dot{\mathcal{E}}$ will be of importance in understanding the van der Pol oscillator.

(4) Nonlinear differential equations need not have well-defined solutions for all time $t$, nor do they have to possess unique solutions. Additional hypotheses are needed to rule out such (perhaps unphysical) behaviors.

has the solution

$$x(t) = \tan(\tan^{-1}(x_0) + (t-t_0))$$

which satisfies $x(t_0) = x_0$, but blows up in finite time (for $t = k\frac{\pi}{2} + t_0 - \tan^{-1}(x_0)$, $k$ integer).

The scalar equation

$$\dot{x}(t) = 3(x(t))^{2/3} \; ; \; x(0) = 0$$

has a whole family of solutions,

$$x_\alpha(t) = \begin{cases} (t-\alpha)^3 & t \geq \alpha \\ 0 & t < \alpha \end{cases}$$

parametrized by $\alpha \geq 0$.

(5) The example of the Lorenz equations (see E. N. Lorenz (1963), "Deterministic non-periodic ~~soluti~~ flows", J. Atm. Sci, 20: 130-141) shows that in dimension 3 (and higher) one sees new dynamical phenomena in ordinary

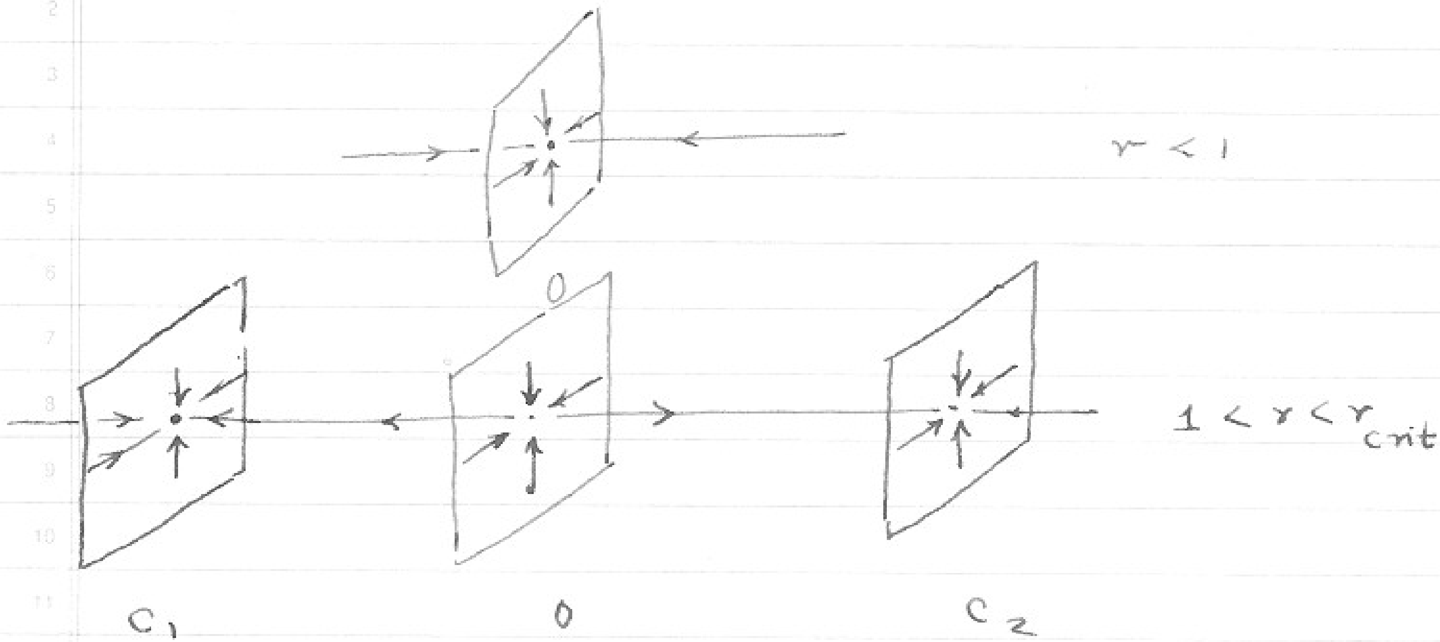
periodic orbits. In $\mathbb{R}^2$, one can also observe limit cycles. In $\mathbb{R}^3$, we have room for greater complexity.

The Lorenz equations, arising from an approximate model of convective heat transfer in fluids are:

$$\dot{x} = \sigma \cdot (y - x)$$
$$\dot{y} = r \cdot x - y - xz$$
$$\dot{z} = xy - bz.$$

Here $x$ corresponds to convective overturning in a 2-D fluid cell, $y$ corresponds to horizontal temperature variation and $z$ corresponds to vertical temperature variation. The parameter $\sigma$ is proportional to the Prandtl number, $r$ the Rayleigh number and $b$ is a length scale.

For $0 < r < 1$, the origin is the <u>only</u> equilibrium point and it is globally <u>attracting</u>. $r = 1$ is a <u>bifurcation point</u>. For $r > 1$, one obtains two additional equilibria (besides 0). For this parameter range, the origin becomes unstable and the other two equilibria (denoted $C_1$ and $C_2$ in Figure 5) are stable provided




$x = -\sqrt{b(r-1)}$

$y = x$

$z = \frac{x^2}{b}$

$x = +\sqrt{b(r-1)}$

$y = x$

$z = \frac{x^2}{b}$

Figure 5

For $r > r_{crit}$ complicated dynamics emerge. For $\sigma = 10$, $b = \frac{8}{3}$, $r_{crit} = 24.74$. Consider say $r = 28$. The solution is not periodic, not transient, but complicated

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The examples discussed above should convey an idea of the exceedingly interesting range of phenomena in nonlinear systems. Nonlinear control theory seeks to manipulate such phenomena in a controlled way, and this is a task that requires understanding the phenomena as well as the mathematical structures and techniques that allow one to solve such synthesis and design problems.