ENEE 661 Nonlinear Control Systems

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Lecture 4 (part i)

In this lecture, we discuss the existence and uniqueness of solutions to ordinary differential equations. The central idea is the Contraction Mapping – Fixed Point Theorem due to S. Banach.

Theorem 1: Let X be a Banach space and let $S \subset X$ be a closed subset. Let $f : S \Rightarrow S$ be a mapping such that, for some $0 \le \rho < 1$,

$$\| f(x) - f(y) \| \le \rho \, \| x - y \| \qquad \forall \; x, y \in S$$

(such a map is called a contraction.)

Then there is a unique $x^* \in S$ such that $x^* = f(x^*)$. Further this fixed point can be obtained by the method of successive approximations (Banach iterations).

Proof: Let $x_1 \in S$. Define the sequence $\{ x_k : k \geq 1 \}$ by

$$x_{k+1} = f(x_k).$$

By hypothesis, $\{x_k\} \subset S$.

Note: Stefan Banach was a central figure in the mathematical life of Poland in pre-WW II era
see: http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Banach.html

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$$\| x_{k+1} - x_k \| = \| f(x_k) - f(x_{k-1}) \|$$

$$\leq \rho \| x_k - x_{k-1} \|$$

$$\leq \rho^2 \| x_{k-1} - x_{k-2} \|$$

(repeating the previous step)

$$\vdots$$

$$\leq \rho^{k-1} \| x_2 - x_1 \|$$

Hence, $\| x_{k+r} - x_k \| \leq \| x_{k+r} - x_{k+r-1} + x_{k+r-1} \cdots -$

$$\leq \| x_{k+r} - x_{k+r-1} \| + \| x_{k+r-1} - x_{k+}$$

$$+ \cdots + \| x_{k+1} - x_k \|$$

$$\leq (\rho^{k+r-2} + \rho^{k+r-3} + \cdots + \rho^{k-1}) \| x_2 -$$

for $k \geq 1$

$$\leq \rho^{k-1} \sum_{j=0}^{\infty} \rho^j \cdot \| x_2 - x_1 \|$$

$$= \frac{\rho^{k-1}}{1-\rho} \| x_2 - x_1 \|$$

$\rightarrow 0$ as $k \rightarrow \infty$, since $\rho <$

Hence $\{x_k\}$ is a Cauchy sequence. Since X is a Banach space there is x^* such that $x_k \rightarrow x^*$. But S is closed. Therefore $x^* \in S$. To see that x^* is a

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fixed point,

$$\|x^* - f(x^*)\| \le \|x^* - x_k\| + \|x_k - f(x^*)\|$$
$$\le \|x^* - x_k\| + \rho \|x_{k-1} - x^*\|$$
$$\to 0 \quad \text{as } k \to \infty.$$

Hence $\|x^* - f(x^*)\| = 0 \Rightarrow x^* = f(x^*)$.

To prove **uniqueness**, suppose $y^* \in S$ is another fixed point.

$$\|x^* - y^*\| = \|f(x^*) - f(y^*)\|$$
$$< \rho \|x^* - y^*\|$$

But $\rho < 1$. So $\|x^* - y^*\| = 0 \Rightarrow x^* = y^*$.

If the mapping f were to depend on a parameter in a continuous way, so does the fixed point.

<u>Theorem 2</u> [Continuity of Fixed Point w.r.t. Parameter

Let Θ be a metric space with metric d. Let X be a Banach space and let $S \subset X$ be a closed subset, such that

$$f : \Theta \times S \to S$$

has the following properties :

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(i) Each partial map ~~[illegible]~~

$$f_\theta : S \to S \qquad \theta \in \Theta$$

(defined by $f_\theta(x) = f(\theta, x)$), is a contraction with parameter $\rho < 1$.

(ii) For each $x \in S$, the partial map

$$f_x : \Theta \to S$$

(defined by $f_x(\theta) = f(\theta, x)$), is continuous i.e. given $\epsilon > 0$ there exists $\delta_x > 0$ s.t. $d(\theta, \theta') < \delta_x \Rightarrow \| f_x(\theta) - f_x(\theta') \| < \epsilon$

Then, if x^*_θ is the unique fixed point of f_θ, the map $\theta \mapsto x^*_\theta$ is continuous.

Proof: $\| x^*_\theta - x^*_{\theta'} \| = \| f_\theta(x^*_\theta) - f_{\theta'}(x^*_{\theta'}) \|$

$$\leq \| f_\theta(x^*_\theta) - f_\theta(x^*_{\theta'}) \| + \| f_\theta(x^*_{\theta'}) - f_{\theta'}(x^*_{\theta'}) \|$$

$$\leq \rho \| x^*_\theta - x_{\theta'} \| + \| f_{x^*_{\theta'}}(\theta) - f_{x^*_{\theta'}}(\theta') \| .$$

Hence

$$\| x^*_\theta - x^*_{\theta'} \| \leq \frac{1}{1-\rho} \| f_{x^*_{\theta'}}(\theta) - f_{x^*_{\theta'}}(\theta') \|$$

$$< \frac{\epsilon}{1-\rho} \quad \text{whenever } d(\theta, \theta') < \delta_{x^*_\theta}$$

This proves continuity of the fixed point.

<u>Example</u> (Jacobi's algorithm)

The linear equation in $\mathbb{R}^n$,

$$Ax = b$$

where A is a square matrix can be identified as the fixed-point problem

$$x = -D^{-1}(L+U)x + D^{-1}b$$

where, $A = L + D + U$ denotes the decomposition into strictly lower triangular, diagonal, and strictly upper triangular parts and we assume D is invertible.

Jacobi's algorithm, to solve this problem:

$$x_{k+1} = -D^{-1}(L+U)x_k + D^{-1}b$$

is a special case of the Banach iteration and, to guarantee convergence, it is sufficient that A be diagonally dominant:

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

Then we can take $\rho = \frac{1}{|a_{ii}|} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \right)$,

making $f(x) = -D^{-1}(L+U)x + D^{-1}b$, a contraction on all of $\mathbb{R}^n$.

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Example

Consider the scalar equation

$$g(x) = x^2 - b = 0 \qquad b > 0.$$

Let $y = 1 - x$. The problem of finding the (positive) square root of b is a fixed-point problem

$$y = \frac{1}{2}\left[(1-b) + y^2\right] = f(y).$$

Suppose $|1-b| < \rho < 1$.

Then f maps the **closed** subset $S = \{ y : |y| \leq \rho \} \subset \mathbb{R}$ into itself and it is a contraction on S with parameter ρ. Thus the algorithm

$$y_{n+1} = \frac{1}{2}\left[(1-b) + y_n^2\right]$$

Converges for $|1-b| < \rho < 1$.

It is equivalent to

$$x_{n+1} = x_n - \frac{1}{2} x_n^2 + \frac{1}{2} b.$$

How does it compare with Newton's method?

We are interested in (and ready for applying) Banach's theorem to o.d.e's.

Let $\dot{x} = f(t, x)$ be a **non-autonomous** ordinary differential equation. A continuously differentiable

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function $x(t)$ is a solution if,

$$\boxed{x(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma}$$

$t \in [t_0, t_0+\delta]$, for some $\delta > 0$. We aim to show existence and uniqueness of solutions to the above integral equation in a suitable function space, the space $(X, \|\cdot\|_X)$ below.

For any $\delta > 0$, the space

$$X = \{\psi : [t_0, t_0+\delta] \to \mathbb{R}^n \mid \psi \text{ continuous}\}$$

with norm

$$\|\psi\|_X = \max_{t \in [t_0, t_0+\delta]} \|\psi(t)\|$$

where $\|\cdot\|$ in $\mathbb{R}^n$ is <u>any</u> norm, is a complete normed linear space, i.e. Banach space. (<u>Proof of completeness</u> $\rightarrow$ <u>exercise</u>)

See appendix B, Example B.1 (Khalil 3rd ed.)

<u>Theorem 3</u> (Local Existence and Uniqueness)

Consider the system

$$\dot{x}(t) = f(t, x),$$

Let f be piecewise continuous in t and satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq L \|x - y\|$$

$\forall\ x, y \in B_r(x_0) = \{x : \|x - x_0\| \leq r\}$ and $\forall\ t \in [t_0, t_1]$. ~~Then~~ there is some $\delta > 0$ such that the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma \qquad t \in [t_0, t_0+\delta]$$

$B_r(x)$

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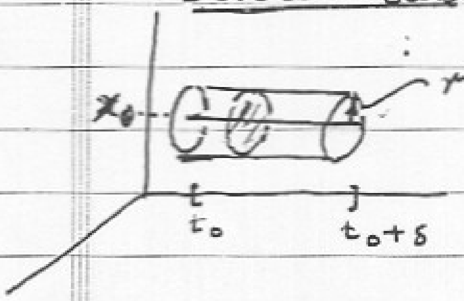
has a unique solution x in X. It is differentiable with respect to t and $\dot{x}(t)$ agrees with $f(t, x(t))$ at all points of continuity in t of f.

<u>Proof</u>: Define $P : X \to X$

$$(Px)(t) = x_0 + \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma \qquad t \in [t_0, t_0+\delta]$$

Let $x_0(\cdot)$ denote the constant function belonging to X, $x_0(t) \equiv x_0$ for $t \in [t_0, t_0+\delta]$.

Let $S := \{ x \in X \mid \|x - x_0\|_X \le r \}$ the <u>solid tube</u> in the figure. It is a <u>closed ball</u> in X.



We will be choosing $\delta > 0$ such that $t_0 + \delta \le t_1$

<u>Observation (i)</u> Since f is piecewise continuous in t, so is $\|f(t,x)\|$ for every x. Thus $\|f(t, x_0)\|$ is bounded on $[t_0, t_1]$. We set

$$h = \max_{t \in [t_0, t_1]} \| f(t, x_0) \| .$$

<u>Observation (ii)</u> Let $x(\cdot) \in S$. Then for $t \le t_0 + \delta$

$$\|(Px)(t) - x_0\| = \left\| \int_{t_0}^{t} f(\sigma, x(\sigma))\, d\sigma \right\|$$

$$\le \int_{t_0}^{t} \| f(\sigma, x(\sigma)) \|\, d\sigma$$

(we use triangle inequality here)

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$$\leq \int_{t_0}^{t} \| f(\sigma, x(\sigma)) - f(\sigma, x_0) \| \, d\sigma + \int_{t_0}^{t} \| f(\sigma, x_0) \| \, d\sigma$$

(triangle inequality)

$$\leq \int_{t_0}^{t} \left(L \| x(\sigma) - x_0 \| + h \right) d\sigma$$

(by Lipschitz condition & obs (i))

$$\leq \int_{t_0}^{t} (L \cdot r + h) \, d\sigma \qquad (\text{since } x(\cdot) \in S)$$

$$= (t - t_0)(Lr + h)$$

$$\leq \delta \cdot (Lr + h).$$

Hence $\| Px - x_0 \|_X = \max_{t_0 \leq t \leq t_0 + \delta} \| (Px)(t) - x_0 \|$

$$\leq \delta \cdot (Lr + h)$$

$$\leq r, \text{ if } \delta \leq \frac{r}{Lr + h}.$$

So choosing $\delta \leq \frac{r}{Lr+h}$ ensures that P maps S into S.

In this case,

Observation (iii) P is a contraction on S.

To see this, let $x, y \in S$.

$$\| (Px)(t) - (Py)(t) \| = \left\| \int_{t_0}^{t} [f(\sigma, x(\sigma)) - f(\sigma, y(\sigma))] \, d\sigma \right\|$$