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$$\leq \int_{t_0}^{t} \| f(\sigma, x(\sigma)) - f(\sigma, y(\sigma)) \| d\sigma$$

$$\leq \int_{t_0}^{t} L \cdot \| x(\sigma) - y(\sigma) \| d\sigma \qquad \text{(by Lipschitz condition)}$$

$$\leq L \cdot (t - t_0) \; \| x(\cdot) - y(\cdot) \|_X$$

Hence $\| Px - Py \|_X \leq L \cdot \delta \; \| x - y \|_X$

$$\leq \rho \; \| x - y \|_X ,$$

if $\delta \leq \frac{\rho}{L}$

Thus choosing $\rho < 1$, and

$$\delta \leq \min \left(t_1 - t_0 , \; \frac{r}{Lr + h} , \; \frac{\rho}{L} \right)$$

ensures that $P : S \to S$ is a contraction mapping.

Hence, by the theorem of Banach there is a unique fixed point for P in S, the solution to the integral equation. We can actually show that this is the only solution in X.

Since $x_0 \in B(x_0, r)$, any (continuous) solution $x(t)$ must ~~leave~~ lie inside $B(x_0, r)$

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for some nontrivial interval of time. Suppose $x(t)$ leaves $B(x_0, r)$ and $t_0+\mu$ is the first instant of time that $x(t)$ intersects $\partial B(x_0, r)$ the boundary of $B(x_0, r)$. Then

$$\| x(t_0+\mu) - x_0 \| = r.$$

On the other hand, $\forall\ t \le t_0+\mu$,

$$\| x(t) - x_0 \| \le \int_{t_0}^{t} (Lr+h)\, ds \qquad \text{(see obs(ii))}$$

so that $\quad r = \| x(t_0+\mu) - x_0 \|$

$$\le (Lr+h)\mu \Rightarrow \mu \geq \frac{r}{Lr+h} \geq \delta.$$

Hence the solution starting at x_0 stays in $B(x_0, r)$ and hence in S during $[t_0, t_0+\delta]$. Consequently uniqueness of solution in S $\Rightarrow$ uniqueness in X. ▨

$\boxed{\text{Here Banach iteration} = \text{Picard–Lindelöf iteration}}$

Notice that the map P in the local existence and uniqueness theorem depends on x_0 in a continuous way.

Corollary 4 Let $\textcircled{H}$ be a metric space that parametrizes a family of differential equations and initial conditions. Suppose the parametrization is such that the conditions of Theorem 2 are satisfied. Then by Theorem 2 the solutions

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obtained in Theorem 3 depends continuously on x_0 and more generally $\theta \in$ (4). ∎

Corollary 4 is a very useful result to keep in mind. The following lemma leads to comparison of solutions

Gronwall-Bellman Inequality / Lemma 5

Let $\lambda : [a,b] \to \mathbb{R}$ be continuous and $\mu : [a,b] \to \mathbb{R}$ be continuous and non-negative. If a continuous function $y : [a,b] \to \mathbb{R}$ satisfies

(implicit)

$$y(t) \le \lambda(t) + \int_a^t \mu(s) y(s)\, ds, \qquad a \le t \le b,$$

then

(explicit)

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau)d\tau\right] ds, \qquad a \le t \le b.$$

In particular if $\lambda(t) \equiv \lambda$ is a constant, then

$$y(t) \le \lambda \exp\left[\int_a^t \mu(\tau)d\tau\right].$$

If in addition, $\mu(t) \equiv \mu \geq 0$ is a constant, then

$$y(t) \le \lambda \exp[\mu(t-a)]$$

Proof: Let $z(t) = \int_a^t \mu(s) y(s)\, ds$ and

$$v(t) = z(t) + \lambda(t) - y(t) \geq 0.$$

Then, z is differentiable and

$$\dot{z}(t) = \mu(t) y(t) = \mu(t) z(t) + \mu(t)\lambda(t) - \mu(t)v(t).$$

This scalar equation has the solution, (since $z(a) = 0$)

$$z(t) = \int_a^t \phi(t,s)\left[\mu(s)\lambda(s) - \mu(s)v(s)\right] ds$$

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where $\phi(t,s) = \exp\left[\int_s^t \mu(\tau)\,d\tau\right] > 0$

By hypothesis, $\int_a^t \phi(t,s)\,\mu(s)\,v(s)\,ds \geq 0$. Therefore,

$$z(t) \leq \int_a^t \exp\left(\int_s^t \mu(\tau)\,d\tau\right) \cdot \lambda(s)\mu(s)\,ds$$

and, since $y(t) \leq \lambda(t) + z(t)$, the proof of the general case is completed.

The remaining cases amount to computing integrals — left to the reader. ∎

Corollary 6 $f(t,x)$ is piecewise continuous in t and Lipschitz in x on $[t_0, t_1] \times W$ with Lipschitz constant L, where $W \subset \mathbb{R}^n$ is an open connected set. Let $y(t)$ and $z(t)$ be solutions of

$$\dot{y} = f(t,y);\quad y(t_0) = y_0$$

and

$$\dot{z} = f(t,z) + g(t,z);\quad z(t_0) = z_0$$

such that $y(t), z(t) \in W \;\forall\, t \in [t_0, t_1]$.

Suppose

$$\|g(t,x)\| \leq \mu \quad \forall\, (t,x) \in [t_0, t_1] \times W$$

for some $\mu \geq 0$, and

$$\|y_0 - z_0\| \leq \gamma.$$

Then,

$$\|y(t) - z(t)\| \leq \gamma \exp[L(t-t_0)] + \frac{\mu}{L}\left\{e^{L(t-t_0)} - 1\right\}$$
$$\forall\, t \in [t_0, t_1]$$

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Proof : $\quad y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$

$$z(t) = z_0 + \int_{t_0}^{t} \left[f(s, y(s)) + g(s, z(s)) \right] ds .$$

Then,

$$\| y(t) - z(t) \| \le \| y_0 - z_0 \| + \int_{t_0}^{t} \| f(s, y(s)) - f(s, z(s)) \|\, ds$$

$$+ \int_{t_0}^{t} \| g(s, z(s)) \|\, ds$$

$$\le \gamma + \mu \cdot (t - t_0) + \int_{t_0}^{t} L \, \| y(s) - z(s) \|\, ds.$$

By the Gronwall–Bellman inequality,

$$\| y(t) - z(t) \| \le \gamma + \mu \cdot (t - t_0) + \int_{t_0}^{t} L \cdot (\gamma + \mu (s - t_0)) \cdot e^{L \cdot (t-s)}\, ds$$

$$= \gamma + \mu \cdot (t - t_0) - \gamma - \mu \cdot (t - t_0) + \gamma \cdot e^{L \cdot (t - t_0)} + \int_{t_0}^{t} \mu \cdot e^{L \cdot (t-s)}\, ds$$

(integration by parts)

$$= \gamma e^{L \cdot (t - t_0)} + \frac{\mu}{L} \left(e^{L \cdot (t - t_0)} - 1 \right) \quad \blacksquare$$

Remark In these applications as in the original Gronwall–Bellman inequality, one is turning an implicit inequality explicit — in effect "solving the inequality".

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__Remark__

Corollary 6 allows us to quantitatively estimate the effects of perturbations – in initial conditions and the dynamics. Such estimates are useful to keep in mind — all models of physical systems display errors due to various unavoidable approximations.

__Theorem 7__ (Global Existence and Uniqueness)

Suppose $f(t,x)$ in Theorem 3 is piecewise continuous in t, and satisfies

$$\|f(t,x_0)\| \le h$$

global Lipschitz $\rightarrow$ $\|f(t,x)-f(t,y)\| \le L \cdot \|x-y\|$

$$\forall\, x,y \in \mathbb{R}^n,\ t\in[t_0,t_1]$$

then the o.d.e

$$\dot{x}(t) = f(t,x) \quad \text{with } x(t_0)=x_0$$

has a unique solution on $[t_0, t_1]$.

__Proof:__ We show how to modify the proof of Theorem 3. There we now let r be arbitrarily large (due to the global Lipschitz condition) so that $\frac{r}{Lr+h} > \frac{\rho}{L}$.

{by taking $r > \frac{\rho}{(1-\rho)}\frac{h}{L}$)

Thus we only need $\delta \le \min\left\{t_1-t_0, \frac{\rho}{L}\right\}$ for $\rho > 1$. If $t_1 - t_0 \le \frac{\rho}{L}$, we could let $\delta = t_1 - t_0$ and we are done.

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If not, choose $\delta = \frac{\rho}{L}$, divide $[t_0, t_1]$ into finite number of subintervals of length $\delta = \frac{\rho}{L}$ and repeat that many times the arguments of Theorem 3. This completes ~~the proof~~ $\blacksquare$

EXAMPLE

$$\dot{x} = -x^3$$

does not satisfy a global Lipschitz condition. But there is a unique solution

$$x(t) = \operatorname{sgn}(x_0)\sqrt{\frac{x_0^2}{1+2x_0^2\cdot(t-t_0)}} \qquad t \geq t_0$$

through $x(t_0) = x_0$.

The essential idea here is that if $x(0) = a$, the set $\{x : |x| \leq a\}$ is a positively invariant closed and bounded ~~solution~~ set for the dynamics $\dot{x} = -x^3$. This idea can be generalized to

Theorem

$f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz on a domain $D \subset \mathbb{R}^n$. Suppose there is a closed and bounded set $W \subset D$ such that $x(0) = x_0 \in D$ and $f|_{\partial W}$ points into W. Then there is a unique solution $x(t)$ of $\dot{x} = f(x)$ such that $x(0) = x_0$

Proof: EXERCISE.

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Some Definitions and Properties Pertaining to Lipschitz Condition.

open, connected

Def^n (a) f is locally Lipschitz on a domain $D \subset \mathbb{R}^n$ if each ~~point~~ p of D has neighborhood (i.e ball $B_\varepsilon(p)$ surrounding p, $\varepsilon > 0$) such that

$$\|f(x) - f(y)\| \le L_{B_\varepsilon} \cdot \|x - y\|$$

$\forall\ x, y \in B_\varepsilon$ and $L_{B_\varepsilon} > 0$.

Def^n (b) f is Lipschitz on a set W if

$$\|f(x) - f(y)\| \le L \cdot \|x - y\|, \quad \forall x, y \in W.$$

Properties

(a) f is locally Lipschitz on a domain D $\Rightarrow$ f is continuous on D.

(b) f is Lipschitz on domain $D \Rightarrow f$ is uniformly continuous on D

(c) Converse of ~~D~~ (a) is not true:

Consider $f(x) = x^{1/3}$ on $(-1, 1)$

(d) f is locally Lipschitz on domain D $\not\Rightarrow$ f is Lipschitz on D (due to lack of uniformity of the Lipschitz constant

(e) f is locally Lipschitz on domain D $\Rightarrow$ Lipschitz on every closed and bounded subset of D.

(f) f is continuously differentiable $\Rightarrow f$ is locally Lipschitz. Converse is far from true.