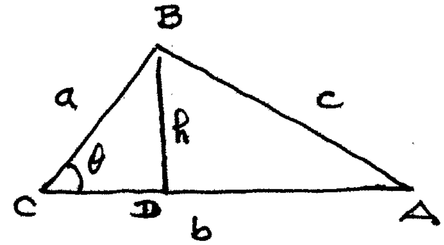
Some computations pertaining to the *index*

1. Area.



The area of a triangle with sides a, b, c is given by adding the areas of the two right triangles, BCD and BAD.

$$\text{Area} = \frac{1}{2} h \cdot CD + \frac{1}{2} h \cdot AD$$

$$= \frac{1}{2} h \cdot b$$

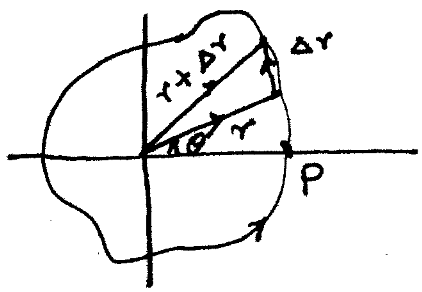
$$= \frac{1}{2} a b \sin(\theta)$$

Recall, from the definition of the vector product, that

$$|\vec{CA} \times \vec{CB}| = |\vec{CA}|\,|\vec{CB}| \sin\theta$$
$$= b \quad a \quad \sin\theta$$

Thus the oriented / signed area of the triangle ABC is $\frac{1}{2} \vec{CA} \times \vec{CB}$

2. Area enclosed by a parametrized closed curve $\gamma : [0, T] \longrightarrow \mathbb{R}^2$, $t \longmapsto \gamma(t)$
$\gamma(0) = \gamma(T) = P$, say.



The enclosed area is obtained by adding up areas of triangles bounded by the vectors $\vec{r}$, Δr and $r + \Delta r$ and taking the limit as $\Delta r \to 0$.

$$\text{area} \approx \sum_{i=1}^{N} \tfrac{1}{2} r_i \times (r_i + \Delta r_i)$$

$$= \sum_{i=1}^{N} \tfrac{1}{2} r_i \times \Delta r_i \qquad \left(\begin{array}{l} \text{since for any } v \\ v \times v = 0 \end{array} \right.$$

Taking the limit as the number of terms in the sum goes to ∞, we get.

$$\text{area} = \oint \tfrac{1}{2} \, r(t) \times \frac{dr(t)}{dt} \, dt$$

$$= \int_0^T \tfrac{1}{2} \, r(t) \times \dot{r}(t) \, dt$$

Let $\quad r(t) = x(t) \, \underline{i} + y(t) \, \underline{j}$

Then $\quad r(t) \times dr(t) = (x \, \underline{i} + y \, \underline{j}) \times (dx \, \underline{i} + dy \, \underline{j})$

$$= (x \, dy - y \, dx) \, \underline{k}$$

where $\underline{k} = \underline{i} \times \underline{j}$ is unit normal to plane spanned orthonoromal basis vectors $\underline{i}$, $\underline{j}$.

Thus $\quad \text{area} = \tfrac{1}{2} \oint (x \, dy - y \, dx) \, \underline{k}$

3. Polar coordinates

Let $x = r\cos(\theta)$; $y = r\sin(\theta)$

Then, $\frac{1}{2}(x\,dy - y\,dx)$

$$= \frac{1}{2}\Big(r\cos\theta\,(r\cos\theta\,d\theta + dr\sin\theta) - r\sin\theta\,(-r\sin\theta\,d\theta + dr\cos\theta)\Big)$$

$$= \frac{1}{2}r^2\,d\theta$$

Signed area enclosed by curve γ

$$= \oint \frac{1}{2}r^2\,d\theta \qquad \underline{k}$$

with curve traversed in the counter clockwise direction.

For γ = unit circle centered at 0,

$$\text{area} = \frac{1}{2}R^2\,2\pi\,\Big|_{R=1}$$

$$= \pi$$

4. $$d\theta = d\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\,d\left(\frac{y}{x}\right)$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\,dy - \frac{y}{x^2}\,dx\right)$$

$$= \frac{x\,dy - y\,dx}{x^2 + y^2}$$

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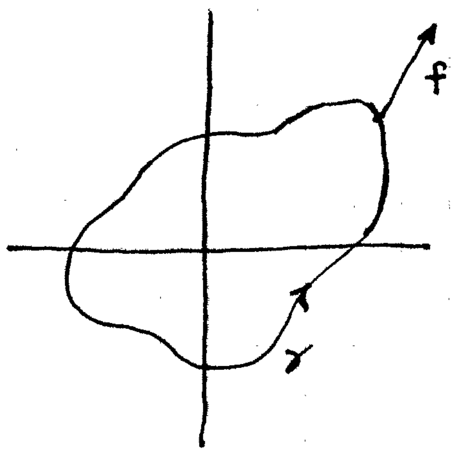


Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field, i.e. the components f_1 and f_2 of f have continuous first partial derivatives.

At any point (x, y) on the plane,

$$\theta_f(x, y) = \tan^{-1}\left(\frac{f_2(x,y)}{f_1(x,y)}\right), \text{ if well-defined.}$$

Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$ be a closed curve, <u>not</u> passing through an equilibrium point of $\dot{x} = f(x)$, i.e. a point such that $f(x) = 0$. We let $\hat{f} = \frac{f}{\|f\|}$ on γ. Then, we have a map

$$\tilde{\gamma} : S^1 \longrightarrow S^1$$

$$t \longmapsto \tan^{-1}\left(\frac{f_2}{f_1}\right)\Big|_{\gamma(t)} = \theta_f\Big|_{\gamma(t)}$$

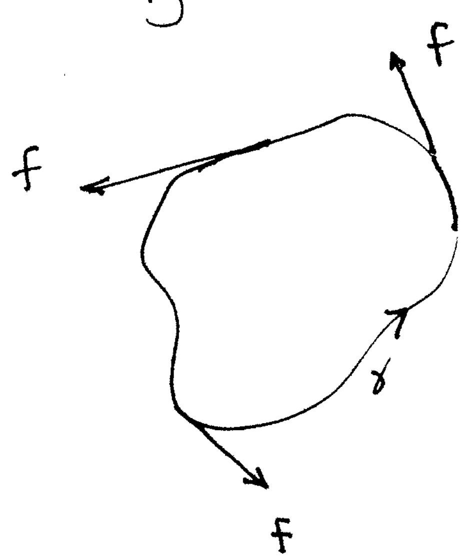
Here S^1 denotes the circle obtained by identifying 0 and 2π.

Then $$\mathrm{Ind}_\gamma^f = \frac{1}{2\pi} \oint d\theta_f$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{d\theta_f}{dt}\right) dt$$

It counts how many times $\tilde{\gamma}$ winds around the circle.

6. Index Ind_γ^f for γ a closed orbit of a vector field.



$$\boxed{f = \frac{d\gamma}{dt}}$$

and $f \neq 0$ on γ.

$$\frac{d\theta_f}{dt} = \frac{d}{dt} \tan^{-1}\left(\frac{f_2}{f_1}\right)$$

$$= \frac{d}{dt} \tan^{-1}\left(\frac{\dot{\gamma}_2}{\dot{\gamma}_1}\right)$$

$$= \frac{\dot{\gamma}_1 \ddot{\gamma}_2 - \dot{\gamma}_2 \ddot{\gamma}_1}{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}$$

The integral

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\dot{\gamma}_1\ddot{\gamma}_2 - \dot{\gamma}_2\ddot{\gamma}_1}{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}\, dt$$

can be computed effectively by a change of variable.

Let $$s(t) = \int_0^t \|\dot{\gamma}(t)\|\, dt = \int_0^t (\dot{\gamma}_1^2 + \dot{\gamma}_2^2)^{1/2}\, dt$$

denote the length of the arc $\{\gamma(\sigma) : 0 \le \sigma \le t\}$. Total length of the closed curve $= s(2\pi)$. By hypothesis that $\dot{\gamma} \neq 0$ on γ, $t \mapsto s(t)$ is a strict monotone increasing function. Hence it can be inverted to obtain $s \mapsto t(s)$. We can think of γ as parametrized by arc-length, by substitution, $s \mapsto \gamma(t(s))$.

Denote by $(\,'\,)$ the derivative operator $\frac{d}{ds}$, and let $v(t) = \sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2}$ be the speed

Then. $$\frac{d}{dt} = \frac{d}{ds}\,\frac{ds}{dt}$$

$$= \nu \frac{d}{ds}$$

Hence $$\dot{\gamma} = \nu \gamma'$$

$$\Rightarrow \quad \dot{\gamma}_1 \ddot{\gamma}_2 - \dot{\gamma}_2 \ddot{\gamma}_1$$

$$= \nu \gamma_1' \left(\gamma_2'' \nu^2 + \gamma_2' \dot{\nu} \right) - \nu \gamma_2' \left(\gamma_1'' \nu^2 + \gamma_1' \dot{\nu} \right)$$

$$= \nu^3 \left(\gamma_1' \gamma_2'' - \gamma_2' \gamma_1'' \right)$$

$$\mathrm{Ind}_\gamma^f = \frac{1}{2\pi} \int_0^{2\pi} \left(\gamma_1' \gamma_2'' - \gamma_2' \gamma_1'' \right) \frac{\nu^3}{\nu^2} \, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\gamma_1' \gamma_2'' - \gamma_2' \gamma_1'' \right) \nu \, dt$$

$$= \frac{1}{2\pi} \oint \left(\gamma_1' \gamma_2'' - \gamma_2' \gamma_1'' \right) ds$$

$$= \frac{1}{2\pi} \oint \left(\gamma_1' \underline{i} + \gamma_2' \underline{j} \right) \times \frac{d}{ds}\left(\gamma_1' \underline{i} + \gamma_2' \underline{j} \right) d.$$

$$= \frac{1}{2\pi}\, 2 \,(\text{area enclosed by curve } s \mapsto \gamma'(s))$$

But $\|\gamma'\| = \|\dot{\gamma}\,\frac{1}{v}\|$

$$= \frac{\|\dot{\gamma}\|}{v}$$

$$= \frac{v}{v} \equiv 1,$$

i.e the curve $s \mapsto \gamma'(s)$ is unit circle. Hence it encloses area $\pi \cdot 1$. Hence

$$\operatorname{Ind}_{\gamma}^{f} = \frac{1}{2\pi} \cdot 2 \cdot \pi$$

$$= 1 \qquad \square$$

We have shown that index of a vector field with respect to γ, a closed orbit of f, is 1.

Given two functions (curves)

$$f : [a, b] \to X$$

and $$g : [a, b] \longrightarrow X ,$$

we say that the functions (curves) are <u>homotopic</u> if we can find a mapping

$$F : [0, 1] \times [a, b] \longrightarrow X$$

such that
$$F(0, t) = f(t)$$
$$F(1, t) = g(t) \qquad \forall\ t \in [a, b]$$

Here all functions are required to be continuous and F (if it exists) is called a homotopy. Here we formalize the idea of continuously deforming a function (curve) into another through intermediate functions (curves), $F(s, \cdot)$ where $s \in [0, 1]$.