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ENEE 661 Nonlinear Control Systems
Lecture 3

We continue our study of ways to recognize the existence (or nonexistence) of periodic solutions to planar nonlinear systems. (All systems we study here are C^1 smooth.)

NOTE: f is C^1 smooth if f_1 and f_2 have continuous first partials. More generally, C^k smooth means ... continuous k'th order partials.

Definition Given a vector field f in the plane and a closed, simple curve γ not passing through an equilibrium point of

$$\dot{x} = f(x),$$

the index I_γ^f is simply the total rotation of the vectorfield as we proceed counterclockwise once around the closed curve γ, measured by

$$I_\gamma^f = \frac{1}{2\pi} \oint_\gamma d\theta_f$$

where $\theta_f = \tan^{-1}\left(\frac{f_2}{f_1}\right)$

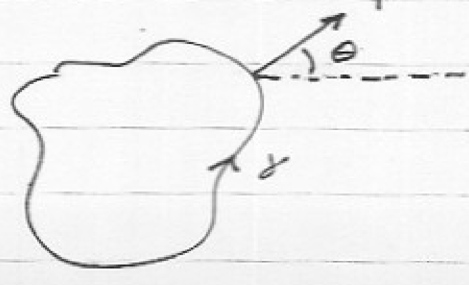

Figure 1

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Since $\frac{d}{dz} \tan^{-1}(z) = \frac{1}{1+z^2}$, it follows that,

$$I_\gamma^f = \frac{1}{2\pi} \oint_\gamma \frac{f_1\, df_2 - f_2\, df_1}{f_1^2 + f_2^2}$$

The index makes its (first) appearance in (H. Poincaré, "Sur les courbes définie par les équations différentielles," Jour. Math. Pures Appl. 4 (1): 167-244, 1885).

Property 1 (homotopy invariance)

A key property of the index is that given two curves γ and γ' in the plane such that γ can be continuously deformed into γ' (or homotoped into γ'), without passing through any equilibria,

$$I_\gamma^f = I_{\gamma'}^f$$

(proof: Since I_γ^f is an integer and it varies continuously as γ is being varied continuously, it does not vary at all as long as we don't cross an equilibrium point)

Property 2

If γ does not enclose any equilibrium points, $I_\gamma^f = 0$.

(proof: by property 1 we can shrink γ to a tiny circle γ' without changing the index. But θ_f is essentially constant on such a circle, thanks to the assumed smoothness of the vector field (as below). So $I_\gamma^f = I_{\gamma'}^f = 0$.)

Figure 2.

Property 3 $\qquad \mathrm{Ind}_\gamma^f = \mathrm{Ind}_\gamma^{-f}$

(proof: use the formula for the index.)

Property 4 If γ is a closed orbit of $\dot{x} = f(x)$ then

$$I_\gamma^f = 1.$$

(proof: by assumption that γ is a closed orbit, the vector field is tangential to γ everywhere on γ as in the adjoining figure. Pick a parametrization with respect to time t of γ.

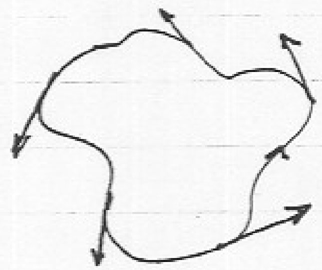
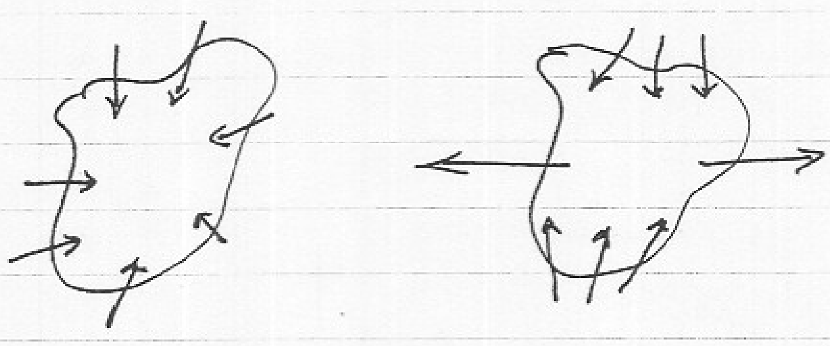


Figure 3.

Then use the formula.)

From the definition of index, the index of γ when it encloses a single node or focus is $+1$ and is -1 when it encloses a single saddle. See Figure

The index of a center is also $+1$. why?



node

saddle

Figure 4

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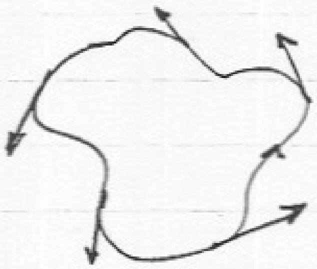

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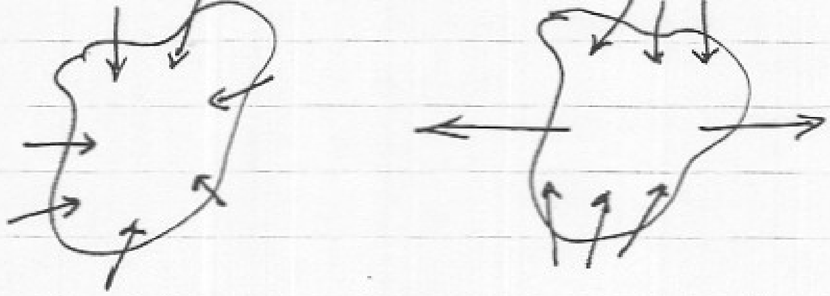

node saddle

Figure 4

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$I(x^*)$

Definition The index of an isolated equilibrium point x^* is defined to be I_γ^f where γ is any closed, simple curve, enclosing x^*

and no other equilibria. (By Property 1 above, this is well-defined, i.e. it is dependent only on x^* and not on the particular γ.)

We can safely say

$$\text{index (node)} = \text{index (focus)} = 1$$
$$\text{index (saddle)} = -1$$

Theorem If a simple closed curve γ enclosed n isolated equilibria $x_1^*, \ldots, x_n^*$ then

$$I_\gamma^f = \sum_{i=1}^{n} I(x_i^*)$$

Corollary A periodic orbit must enclose an equilibrium point.

Question Does this corollary allow one to rule out oscillations in the glycolysis example of Lecture 2 when a, b are such that they lie in the unshaded

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region of Figure 4 (Lecture Notes # 2)?

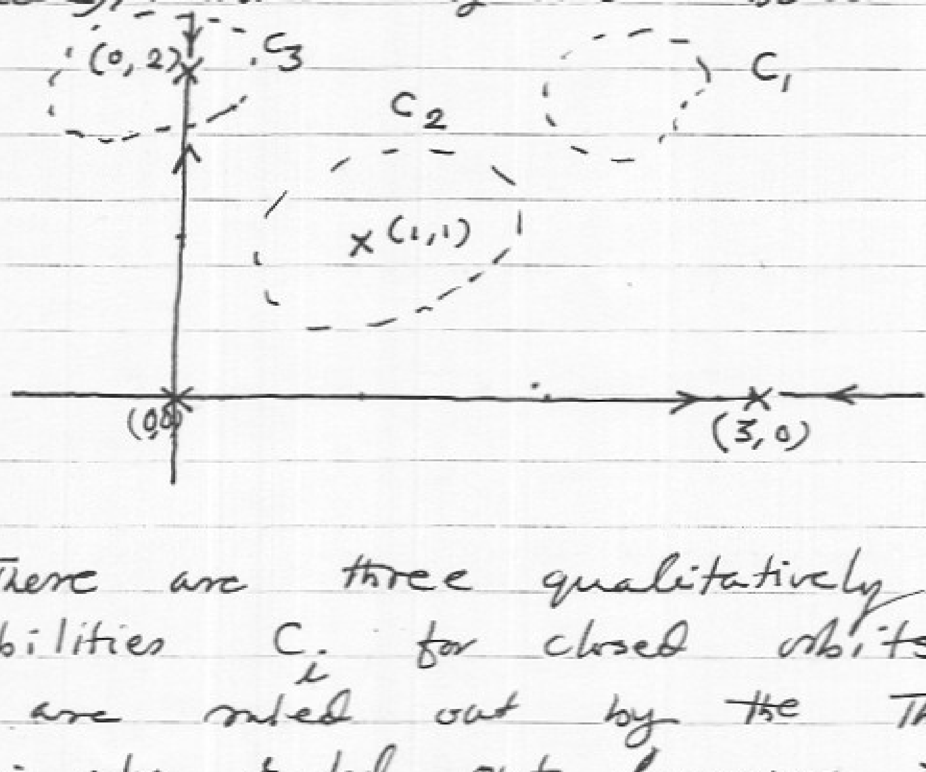
Example. Closed orbits are impossible in the population biology model

$$\dot{x} = x(3-x-2y)$$

$$\dot{y} = y(2-x-y)$$

where $x, y \geq 0$.

Proof Equilibria are $(0,0)$ (unstable node); $(0,2)$, $(3,0)$ (~~un~~stable nodes); and $(1,1)$ (saddle), marked by x's below.



There are three qualitatively distinct possibilities C_i for closed orbits. C_1 and C_2 are ruled out by the Theorem. C_3 is also ruled out because the y axis is a stable manifold & trajectories cannot cross.

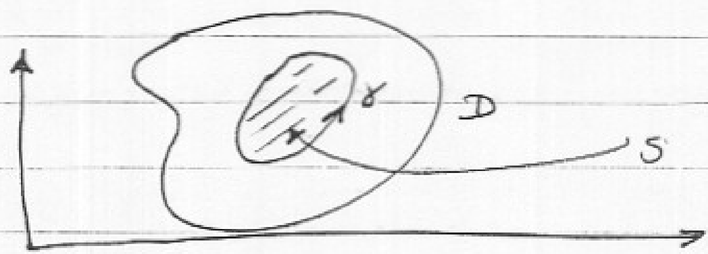
Bendixson's Criterion

Another result that gives us conditions to exclude periodic orbits from certain regions in the plane is attributed to Bendixson.

Let D be a simply connected region in the plane such that $\operatorname{div}(f) \triangleq \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero in any subregion of D and does not change sign in D. Then D does not contain any closed orbits of

$$\dot{x} = f(x) .$$

Proof: Assume, for the sake of contradiction, that γ is a closed orbit in D.



On γ, $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$ implies $\frac{dx_2}{dx_1} = \frac{f_2}{f_1}$, or $f_1\, dx_2 - f_2\, dx_1 = 0$

on γ. Hence $\oint_\gamma f_1\, dx_2 - f_2\, dx_1 = 0$.

But by the theorem of Green this implies

surface integral $\iint_S \operatorname{div}(f)\, dx_1 dx_2 = 0$,

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which contradicts the hypothesis that $\operatorname{div}(f) \not\equiv 0$ and does not change sign on any subregion of D. Hence there can be no periodic orbits γ in D ▨

Remark We can extend this result as Dulac did — observe

$$q\, f_1\, dx_2 - q\, f_2\, dx_1 \equiv 0 \quad \text{on } \gamma$$

and hence $\oint_\gamma q\, f_2\, dx_1 - q f_1\, dx_2 = 0$

$\Rightarrow$ by Green, $\iint_S \operatorname{div}(q f)\, dx_1\, dx_2 = 0$

which would lead to a contradiction if $q(\cdot)$ was picked such $\operatorname{div}(q f) \not\equiv 0$ on any subregion of D. Note,

$$\operatorname{div}(q f) = \nabla q \cdot f + q \operatorname{div}(f),$$

so finding $q(\cdot)$ such that

$$\nabla q \cdot f + q \operatorname{div}(f) > 0 \quad (\text{or } < 0)$$

on D is a problem of solving a partial differential inequality in q.
In practice one tries to "guess" a function $q(\cdot)$.

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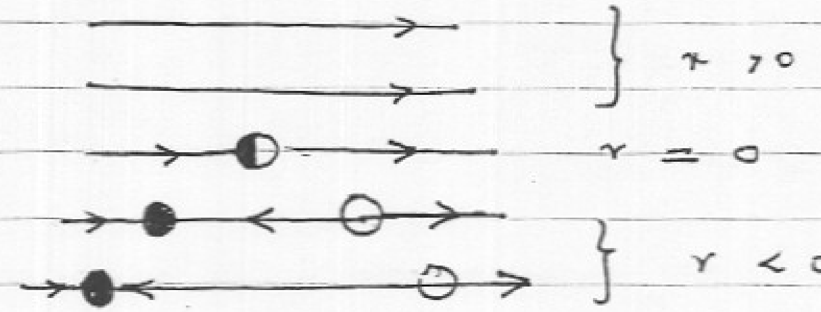
A brief introduction to bifurcations

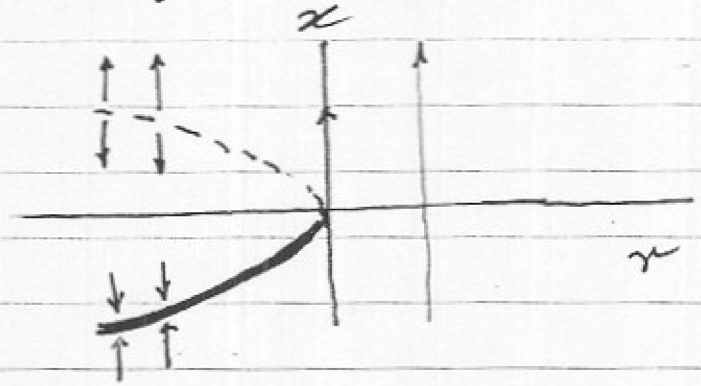
When a differential equation has a settable parameter, the study of how the phase-portrait changes as one continuously varies the parameter is referred to as *bifurcation theory*. The term originates in the *branching* of equilibria as in the Euler buckling problem. We start in dimension 1.

For $\dot{x} = r + x^2$ the phase portraits for different values of r can be 'stacked' in one picture:



The filled circle indicates a stable equilibrium at $x_e = -\sqrt{-r}$ (for $r<0$), while the open circle indicates unstable equilibrium at $x_e = +\sqrt{-r}$ (for $r<0$). The half-filled circle at $x_e = 0$ (for $r=0$) is a "half stable" equilibrium as the directions of arrows suggest.

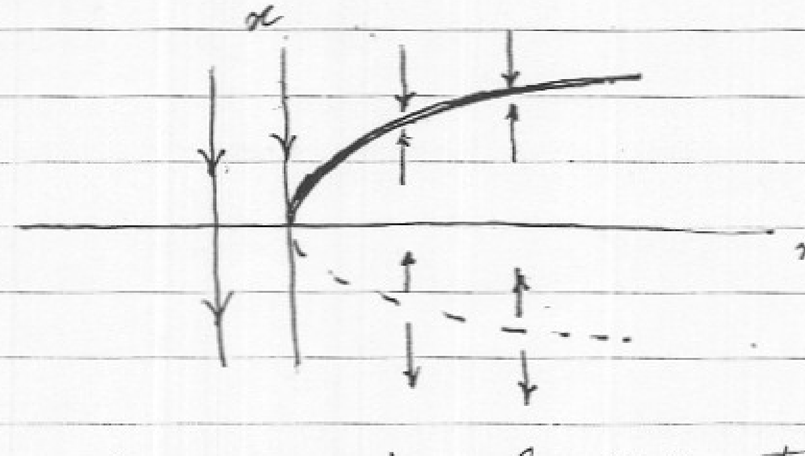
Rotating the above picture $90°$ clockwise and flipping ~~~~ the x axis, we ~~get~~



the above bifurcation diagram. The dotted line indicates an unstable branch (of equilibria), and the solid line a stable branch.

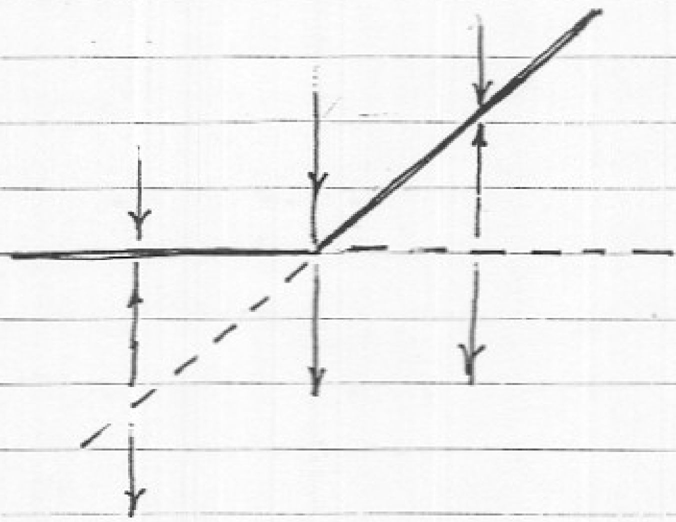
The bifurcation is referred to as the saddle-node (turning point / fold / bluesky bifurcation).

If we had used the form $\dot{x} = r - x^2$ the picture would look like



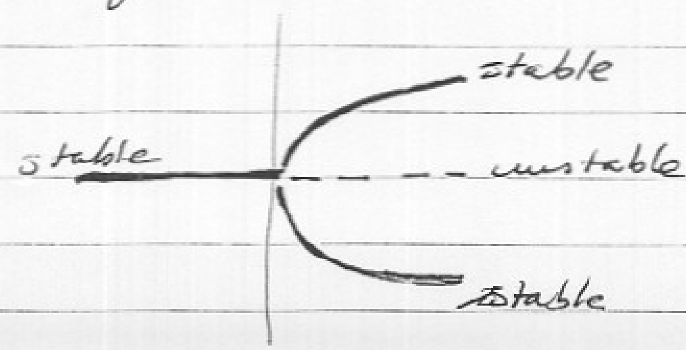
As r increases beyond zero two branches of equilibria ("out of the bluesky") appear. The appearance or disappearance of branches has to do with occurrance of a singularity in $\left(\frac{\partial f}{\partial x}\right)\Big|_{x_e}$ for a critical value of r. One can write down forms which are transcritical (i.e. no change in the number of branches).

Consider $\dot{x} = rx - x^2$. Then $x_e = 0$ is always an equilibrium, so is $x_e = r$. The bifurcation diagram is as below,

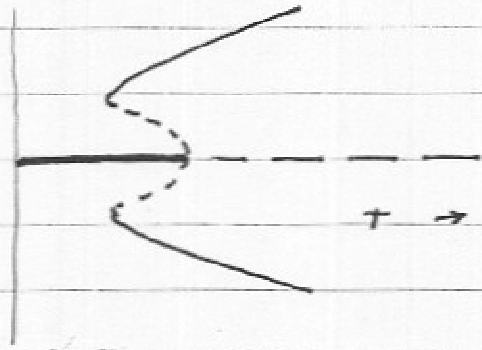


indicating exchange of stability between the two branches.

When the order of the right hand side increases, additional branches appear. For example, $\dot{x} = rx - x^3$ (invariant under $x \to -x$) yields the aptly named supercritical pitchfork bifurcation



Consider $\dot{x} = rx + x^3 - x^5$ with bifurcation diagram (subcritical pitchfork). This suggests possibilities of hysteretic jumps between the $x_e = 0$ stable branch



and the nontrivial stable branches.

One can imbed all these normal forms in 2 dimensions.

Example:

$$\dot{x} = r + x^2$$

$$\dot{y} = -\lambda y .$$

Suppose $\lambda > 0$. Then for $r < 0$, the branch $(x_e = -\sqrt{-r}\,,\ y_e = 0)$ is a branch of (stable) nodes while the branch $(x_e = \sqrt{-r}\,,\ y_e = 0)$ is a branch of saddles. This is the origin of the term saddle-node bifurcation.

We postpone the discussion of dynamic bifurcations such as the Hopf bifurcation. This is the setting in which

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stable limit cycles emerge when an equilibrium at 0 loses stability.