ENEE 661 Nonlinear Control Systems
Lecture 6 (part i)

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The discussion of stability properties in time varying systems is made complicated by the fact that the dependence on initial conditions — specifically initial time, has an effect on how perturbations evolve.
We limit ourselves to the study of stability of equilibria.

Suppose $\tau \mapsto \bar{y}(\tau)$ is a solution to the differential equation

(1) $$\frac{dy}{d\tau} = g(\tau, y) \qquad \tau \geq a.$$

Consider the change of variables,

(2) $$x(t) = y(\tau) - \bar{y}(\tau)$$
$$t = \tau - a$$

Then

$$\dot{x} \triangleq \frac{dx}{dt} = \frac{dx}{d\tau} \cdot \frac{d\tau}{dt}$$

$$= \left(\frac{dy}{d\tau} - \frac{d\bar{y}}{d\tau}\right) \cdot 1$$

$$= g(\tau, y) - \frac{d\bar{y}}{d\tau}$$

$$= g(t+a, x + \bar{y}(t+a)) - \dot{\bar{y}}(t+a)$$

$$\triangleq f(t, x)$$

$$f(t,0) = g(t+a, \bar{y}(t+a)) - \dot{\bar{y}}(t+a)$$
$$= 0, \quad t \geq a \qquad (\text{by } (1))$$

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Thus examining the stability property of the 0-solution of $\dot{x} = f(t, x)$ is equivalent to examining the stability properties of the soln $\bar{y}$ of the equation (1).

Notice that even if g is not explicitly dependent on time, if the solution $\bar{y}$ is non-constant then the transformed system is necessarily non-autonomous due to the term $\dot{\bar{y}}(t+a)$. We are now ready for the basic definitions.

Def 1 (Stability)

The origin $x=0$ is a stable equilibrium for the system $\dot{x} = f(t, x)$ if

(a) $f(t, 0) \equiv 0 \quad t \geqslant 0$

(b) given $\varepsilon > 0$ and any $t_0 \geqslant 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \implies \|\phi(t, t_0, x(t_0))\| \varepsilon$$

$\forall \quad t \geqslant t_0$

(here $\phi(t, t_0, x) =$ soln starting at t_0 at x)

The constant δ in general will depend on t_0. This would mean that if you

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wish to 'trap' the solution in a ball of size ε, starting later, might mean that, one is allowed an even smaller ball of perturbations about 0.

To see this, consider the example:

$$\dot{x} = (6t \sin t - 2t)x$$

This has the solution passing through $x(t_0)$

$$x(t) = x(t_0) \exp\left(\int_{t_0}^{t} (6\sigma \sin\sigma - 2\sigma) d\sigma \right)$$

$$= x(t_0) \exp\left(6\sin(t) - 6t\cos(t) - t^2 - 6\sin(t_0) + 6t_0\cos(t_0) + t_0^2 \right)$$

$$\Rightarrow \quad |x(t)| \leq |x(t_0)| \cdot c(t_0) \qquad t \geq t_0$$

Then for $\varepsilon > 0$ choose $\delta = \dfrac{\varepsilon}{c(t_0)}$

Then $|x(t_0)| < \delta \Rightarrow |x(t)| < \varepsilon$

(STABILITY)

But: $c(t_0) = \exp(-6\sin t_0 + 6 t_0 \cos t_0 + t_0^2 + K)$ where K is a constant, grows with t_0

so δ falls as t_0 increases!

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We need a stronger notion.

Defn 2 (Uniform stability) The equilibrium point 0 of $\dot{x} = f(t,x)$ is uniformly stable if given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ independent of t_0 such that

$$\|x(t_0)\| < \delta \implies \|\phi(t, t_0, x(t_0))\| < \varepsilon$$
$$\forall\ t \geq t_0 \geq 0$$

and the corresponding asymptotic version,

Defn 3 (uniform asymptotic stability) The equilibrium point 0 of $\dot{x} = f(t,x)$ is uniformly asymptotically stable if

(i) it is uniformly stable and

(ii) there exists $c > 0$ independent of t_0, such that (~~the~~ solution ~~starting~~ at $x(t_0)$), $x(t) \to 0$ as $t \to \infty$ uniformly in t_0, $\forall\ \|x(t_0)\| < c$

The device of class $\mathcal{K}$ and class $\mathcal{KL}$ functions (associated with <u>Kamke</u> an Austrian mathematician), is another path to defining notions of stability for time varying systems.

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Aside on class $\mathcal{K}$ functions, class $\mathcal{K}_\infty$ functions $\mathcal{KL}$ functions:

(i) $\alpha : [0, a) \to [0, \infty)$ is class $\mathcal{K}$ if α is continuous, $\alpha(0) = 0$, α is strictly increasing

(ii) $\alpha : [0, \infty) \to [0, \infty)$ is class $\mathcal{K}_\infty$ if it is class $\mathcal{K}$ and $\alpha(x) \to \infty$ as $x \to \infty$ (no levelling off)

(iii) $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is of class $\mathcal{KL}$ if
- (i) β is continuous jointly in the arguments
- (ii) $\beta(\cdot, s)$ is of class $\mathcal{K}$ for each fixed s.
- (iii) $\beta(r, \cdot)$ is decreasing for each fixed r
- (iv) $\beta(r, s) \to 0$ as $s \to \infty$

Examples $\alpha(r) = \tan^{-1}(r)$ is of class $\mathcal{K}$ but not $\mathcal{K}_\infty$.

$\alpha(r) = r^c$, $c > 0$ is of class $\mathcal{K}_\infty$

$\beta(r, s) = r^c e^{-s}$ is of class $\mathcal{KL}$ for $c > 0$

properties α_1, α_2 class $\mathcal{K}$ on $[0, a)$; α_3, α_4 class $\mathcal{K}_\infty$ and $\beta(\cdot, \cdot)$ of class $\mathcal{KL}$. Then the inverse

involution $\alpha_1^{-1} : [0, \alpha_1(a)) \to [0, \infty)$ is of class $\mathcal{K}$

involution α_3^{-1} is of class $\mathcal{K}_\infty$.

algebra $\alpha_1 \circ \alpha_2$ — class $\mathcal{K}$ (functional composition)

algebra $\alpha_3 \circ \alpha_4$ — class $\mathcal{K}_\infty$

$\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is of class $\mathcal{KL}$

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Defn 2' (uniform stability)
The equilibrium point 0 of the timevarying system $\dot{x} = f(t,x)$ is uniformly stable if there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ and a positive constant c independent of t_0 such that

$$\|x(t)\| \le \alpha(\|x(t_0)\|) \quad \forall\, t \geq t_0 \geq 0 \quad \forall\, \|x(t_0)\| < c$$

Defn 3' (uniform asymptotic stability)
The equilibrium point 0 of the timevarying system $\dot{x} = f(t,x)$ is uniformly asymptotically stable if there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a positive constant c independent of t_0 such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0)$$

$$\forall\, t \geq t_0 \geq 0$$
$$\forall\, \|x(t_0)\| < c$$

(further)

Def^n 4' (globally uniformly asymptotically stable)
if

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) \quad \forall\, t \geq t_0 \geq 0$$

~~$\forall\, \|x(t_0)\| \in \mathbb{R}$~~
$\forall\, x(t_0) \in \mathbb{R}^n$

Def^n 5' (exponentially stable) if $\beta(r,s) = k\, r\, e^{-\gamma s}$ $\quad k>0,\ \gamma>0$

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<u>Lemma</u> Definitions (2) and (2)$'$ are equivalent
Definitions (3) and (3)$'$ are equivalent

<u>Proof</u> — left ~~as~~ an exercise.

(<u>hint</u>: for one part of showing 2 & 2$'$ are equivalent, given $\varepsilon > 0$ take $\delta = \alpha^{-1}(\varepsilon)$)

<u>Theorem 1</u> (<u>Uniform asymptotic stability</u>)

Consider the system $\dot{x} = f(t, x)$ satisfying $f(t, 0) \equiv 0 \quad \forall\ t \geq 0$.
Let $D = \{ x \in \mathbb{R}^n \mid \|x\| < r \} = B_r(0)$.
Let $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$

(positive def) (decrescent)

and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f \leq -\alpha_3(\|x\|)$$

$\forall\ t \geq 0$, $\forall\ x \in D$, for three class $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ defined on $[0, r)$.

Then $x = 0$ is uniformly asymptotically stable

(we also assume $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t, locally Lipschitz in x on $[0, \infty) \times D$)

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Proof

Let $\rho < r$. Define,

$$\Omega_{t,\rho} \triangleq \{ x \in B_r(0) \mid V(t,x) \leq \alpha_1(\rho) \}$$

Define $\tilde{\rho} \triangleq \alpha_2^{-1}(\alpha_1(\rho))$.

Now,

$$\|x\| \leq \alpha_2^{-1}(\alpha_1(\rho)) \Rightarrow \alpha_2(\|x\|) \leq \alpha_1(\rho)$$

But, by the decrescence property of V,

$$V(t,x) \leq \alpha_2(\|x\|).$$

Thus

$$\|x\| \leq \alpha_2^{-1}(\alpha_1(\rho)) \Rightarrow V(t,x) \leq \alpha_1(\rho)$$

$$\Rightarrow x \in \Omega_{t,\rho}$$

We have shown,

$$B_{\tilde{\rho}}(0) \triangleq B_{\alpha_2^{-1}(\alpha_1(\rho))}(0) \subseteq \Omega_{t,\rho}$$

Further, $x \in \Omega_{t,\rho}$ i.e., $V(t,x) \leq \alpha_1(\rho)$

$$\Rightarrow \alpha_1(\|x\|) \leq \alpha_1(\rho)$$

(by positive definiteness of V)

$$\Rightarrow \|x\| \leq \rho$$

We have shown

$$\Omega_{t,\rho} \subseteq B_\rho$$

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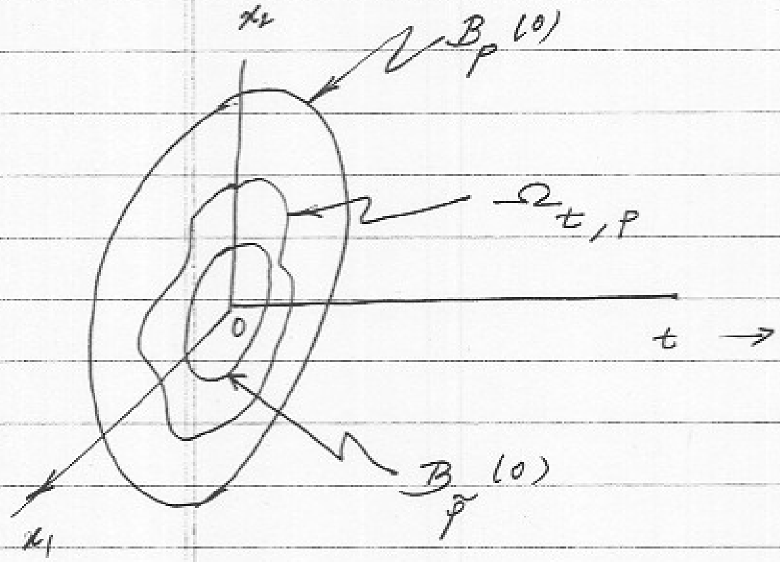
Taken together,

$$\boxed{\mathcal{B}_{\tilde{\rho}} \subset \Omega_{t,\rho} \subset \mathcal{B}_{\rho}} \quad \forall\ t \geq 0$$



We have verified the picture on the left.

Since $\quad \dot{V} = \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x} \cdot f(t, x) < 0$

on $\mathcal{B}_r(0) - \{0\}$ by hypothesis, if $x_0 \in \Omega_{t,\rho}$ the solution starting at (x_0, t_0) stays in $\Omega_{t,\rho}$ $\forall$ $t \geq t_0$.

<u>Assume</u> $\quad x_0 \triangleq x(t_0) \in \mathcal{B}_{\tilde{\rho}}(0)$

By the picture, $\quad x_0 \in \Omega_{t,\rho}$

$$\dot{V} \leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V))$$

$$\triangleq -\alpha(V)$$

(because $\quad V(t, x) \leq \alpha_2(\|x\|)$ by hypothesis).

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The function $\alpha(\cdot)$ defined by

$$\alpha(y) = \alpha_3(\alpha_2^{-1}(y))$$

is class $\mathcal{K}$ on $[0, \alpha_1(\rho))$.

Inspired by the above differential inequality $\dot{V} \leq -\alpha(V)$, we consider the differential equation

$$\dot{y} = -\alpha(y).$$

(we will assume α is locally Lipschitz. if not there exists $\tilde{\alpha}$ locally Lipschitz s.t. $\alpha \geq \tilde{\alpha}$ and we will use $\tilde{\alpha}$ instead of α)

One can prove that there is a class $\mathcal{KL}$ function $\sigma(r, s)$ defined on $[0, \alpha_1(\rho)) \times [0, \infty) \longrightarrow [0, \infty)$ such that

This key step is a technical lemma about scalar differential inequalities (see part (ii) of this lecture) (Lemma 1)

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0)$$

$$\forall V(t_0, x(t_0)) \in [0, \alpha_1(\rho)).$$

Thus for any solution starting at t_0 in $B_{\tilde{\rho}} \subset \Omega_{t_0, \rho}$,

$$\|x(t)\| \le \alpha_1^{-1}(V(t, x(t)))$$
$$\le \alpha_1^{-1}(\sigma(V(t_0, x(t_0)), t-t_0))$$
$$\le \alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|), t-t_0))$$
$$\triangleq \beta(\|x(t_0)\|, t-t_0)$$

and β is class $\mathcal{KL}$ QED

Remark In theorem 1 $B_{\tilde{\rho}}(0)$ is an estimate of the domain of attraction.

Corollary 1 All hypotheses in Theorem 1 are global ($r \to \infty$), and $\alpha_1, \alpha_2 \in K_\infty$. Then 0 is globally uniformly asymptotically stable.

proof $\alpha_2^{-1} \circ \alpha_1 \in K_\infty \Rightarrow \alpha_2^{-1}(\alpha_1(\rho)) \to \infty$ as $\rho \to \infty$

This property is a substitute for radial unboundedness. For any x_0, $\exists\, \rho > 0$ s.t. $\|x_0\| \le \alpha_2^{-1}(\alpha_1(\rho))$. Then the rest of the argument follows as in theorem 1 $\square$

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~~Def~~ Corollary 2 If $\alpha_i(r) = k_i r^c$ $\quad k_i > 0$, $c > 0$.

Then 0 is (uniformly) exponentially stable.

Proof: Track the manipulation of the class $\mathcal{K}$ functions in the proof theorem 1.

$$\alpha(r) = \alpha_3(\alpha_2^{-1}(r))$$

$$= k_3\left(\left(\frac{r}{k_2}\right)^{1/c}\right)^c$$

$$= \frac{k_3}{k_2} r \qquad \text{locally Lipschitz.}$$

$$\Rightarrow \sigma(r,s) = r e^{-\frac{k_3}{k_2}s}$$

$$\Rightarrow \beta(r,s) = \alpha_1^{-1}(\sigma(\alpha_2(r,s), s))$$

$$= \left(\frac{k_2}{k_1} r^c e^{-\frac{k_3}{k_2}s}\right)^{\frac{1}{c}}$$

$$= \left(\frac{k_2}{k_1}\right)^{\frac{1}{c}} r \cdot e^{-\frac{k_3}{ck_2}s}$$

$$\Rightarrow k = \left(\frac{k_2}{k_1}\right)^{\frac{1}{c}}, \quad \gamma = \frac{k_3}{ck_2}$$