ENEE 661: Nonlinear Control Systems

Lecture 8

PSK
04/29/00

Absolute stability — via Lyapunov Theory

This topic originates with the work of Alexander Luré, (1901–?), Russian mathematician from Leningrad/St. Petersburg. The question is easy to state.

Given a linear system with a memoryless nonlinear element in the feedback loop, about which we know very little (say that it lies in a sector), when can we say that the origin is an asymptotically ~~stable equilibrium~~ stable equilibrium for the closed loop system?

[A. I. Luré (1951): Einige nichtlineare Probleme aus der Theorie der automatischen Regelung, Moscow 1951 (R), transl. Berlin (1957)]

Formally,

Given,
$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -\psi(t, y)$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$.

and ψ satisfies, for each $t \geq 0$,

(Sector condition)
$$(\psi(t,y) - K_{min}\, y)^T (\psi(t,y) - K_{max}\, y) \leq 0$$

for K_{min}, K_{max} s.t., $K = K_{max} - K_{min}$ is symmetric and positive definite.

Under what conditions on $G(s) = C(s\mathbb{1} - A)^{-1}B$, K_{min} and K_{max}, can we conclude that the origin is an asymptotically stable equilibrium for the closed loop system.

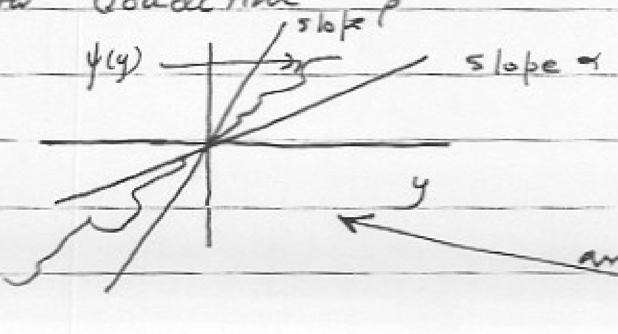
We will treat this problem in a 2 step process. First we restrict to A Hurwitz and $K_{min} = 0$. Then we consider the original question. A bit of terminology — <u>sector condition</u> — can be explained by the following lemma.

<u>Lemma</u> $$\alpha y^2 \le y\,\psi(y) \le \beta y^2 \qquad \alpha \le \beta$$
is equivalent to
$$(\psi(y) - \alpha y)(\psi(y) - \beta y) \le 0$$

<u>Proof</u> ($\Longrightarrow$) suppose $\alpha y^2 \le y\,\psi(y) \le \beta y^2$.
Then $y(\psi(y) - \beta y) \le 0$ and $y(\psi(y) - \alpha y) \ge 0$
Multiplying these two inequalities,
$$y^2 (\psi(y) - \beta y)(\psi(y) - \alpha y) \le 0 .$$
But $y^2 \ge 0$.
Hence $(\psi(y) - \beta y)(\psi(y) - \alpha y) \le 0$

($\Longleftarrow$) Multiply the last inequality by y^2 and reverse/retrace the steps ▨

<u>Remark</u>. The scalar condition
$$(\psi(y) - \alpha y)(\psi(y) - \beta y) \le 0$$
is thus seen to be just the graphical sector condition



Then the condition $(\psi(y) - K_{min}\,y)^T(\psi(y) - K_{max}\,y) \le 0$ is just a multivariable analog of the picture.

Definition Consider the system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -\psi(t,y)$$

where $\forall\, t \geq 0$, $\forall\, y \in \Gamma \subseteq \mathbb{R}^p$, $\mathring{\Gamma}$ connected, $\ni 0$,

$$(\psi(t,y) - K_{min}\, y)^T (\psi(t,y) - K_{max}\, y) \leq 0$$

and $K = K_{max} - K_{min} = K^T > 0$.

The system is absolutely stable with a finite domain Γ if ~~the~~ ~~origin of~~ for the closed loop system, the origin is uniformly, asymptotically, stable with ~~a finite domain~~ any ψ satisfying the sector condit

If $\Gamma = \mathbb{R}^p$, absolute stability $\Leftrightarrow$ global uniform asymptotic stability.

$\mathring{\Gamma}$ = interior of Γ

~~The~~ The main result is the (multivariable) circle criterion. The idea of the proof is to show that under suitable hypotheses one has a time-independent quadratic Lyapunov function. The key ideas here have to do with the concept of passivity.

Recall that mechanical systems without friction can be cast in the hamiltonian form

$$\dot{x} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial x} + f$$

Here f is an external (generalized) force corresponding to the (generalized) coordinate x. Now we define the rate at which mechanical work is done by the external force applied to the system as,

$$\langle f, \dot{x} \rangle .$$

Then

$$\frac{dH}{dt} = \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p}$$

$$= \frac{\partial H}{\partial x} \cdot \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p} \cdot \left(- \frac{\partial H}{\partial x} + f \right)$$

$$= \frac{\partial H}{\partial p} \cdot f$$

$$= \dot{x} \cdot f$$

$$= \text{rate of work } (= \text{power}).$$

So energy stored in the system (ie H) increases at a rate = power input.

If there is internal dissipation then

$$\frac{dH}{dt} \le \text{power},$$

the dissipation inequality.

A passive system is one that satisfies

the dissipation inequality. Treating forces as inputs and (generalized) velocities as outputs, we write the dissipation inequality as

$$H(x(t), p(t)) \leq H(x(0), p(0)) + \int_0^t y^T(\sigma) f(\sigma)\, d\sigma$$

Definition A system is passive if

$$\int_0^{\tilde{T}} y^T(\sigma) \underbrace{f(\sigma)}_{\text{input}}\, d\sigma \geq 0 \qquad \forall\, \tilde{T} \geq 0.$$

This definition is an abstract one for the general setting of input-output systems. For hamiltonians that are a priori bounded below (say $H(x, p) \geq c$) we see that

$$0 \leq \cancel{H(x)}\, H(x(t), p(t)) - c$$
$$\leq H(x(0), p(0)) - c + \int_0^t y^T(\sigma) f(\sigma)\, d\sigma$$

$$\leq \delta + \int_0^t y^T(\sigma) f(\sigma)\, d\sigma$$

which says

$$\int_0^t y(\sigma)^T f(\sigma) \quad \text{is bounded below}$$

(by a constant $-\delta$ that depends on initial conditions), $\forall\, t \geq 0$.

Definition A $p \times p$ matrix $Z(s)$ of transfer functions is positive real if

$Z(s)$ has all matrix elements analytic in $\{s;\ Re(s) \geqslant 0\}$

$Z^*(s) = Z(s^*)$ for $\{s: Re(s) > 0\}$ and

$Z^T(s^*) + Z(s)$ is positive semidefinite for $\{s: Re(s) > 0\}$, where $(*)$ denotes complex conjugation and superscript T denotes matrix transpose.

$Z(s)$ is strictly positive real if $Z(s-\varepsilon)$ is positive real for some $\varepsilon > 0$.

Remark Positive real transfer functions are impedance or admittance matrices made of linear resistors, capacitors, inductors, transformers and gyrators.

Lemma (Kalman - Yacubovitch - Popov)

Let $Z(s) = C(s\mathbb{1} - A)^{-1}B + D$ be the $p \times p$ transfer function of the system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where A is Hurwitz, (A, B) is controllable

(A, C) is observable. Then Z is strict positive real iff there exists $P = P^T > 0$ and matrices W, L and constant $\varepsilon > 0$ such that

$$A^T P + PA = -L^T L - \varepsilon P$$

$$PB = C^T - L^T W$$

$$W^T W = D + D^T$$

Proof (sufficiency)

Suppose there exist P, L, W, ϵ satisfying the above equations. Take $\mu \in (0, \frac{\varepsilon}{2})$.

$$(A + \mu \mathbb{1})^T P + P(A + \mu \mathbb{1}) = -L^T L - (\varepsilon - 2\mu) P$$

$P > 0$ and $L^T L + (\varepsilon - 2\mu) P > 0$. Then by standard matrix Lyapunov theory (e.g. see Theorem 5.36 Lyapunov Lemma on page 211 of Sastry), the matrix $(A + \mu \mathbb{1})$ is Hurwitz. Hence $Z(s-\mu)$ is analytic in $\{s : Re(s) \geq 0\}$.

Let $\Phi(s) = (s\mathbb{1} - A)^{-1}$

$$Z(s-\mu) + Z^T(-s-\mu) = D + D^T + C\Phi(s-\mu)B + B^T \Phi^T(-s-\mu) C^T$$

Substituting $C = (PB + L^T W)^T$ and $D + D^T = W^T W$ we get,

$$Z(s-\mu) + Z^T(-s-\mu) = W^T W + (B^T P + W^T L)\Phi(s-\mu)B + B^T \Phi^T(-s-\mu)(PB + L^T W)$$

$$= W^T W + W^T L \Phi(s-\mu) B + B^T \Phi^T(-s-\mu) L^T W$$
$$+ B^T \Phi^T(-s-\mu) P B$$
$$+ B^T P \Phi(s-\mu) B$$

← Fill in the rest →

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Theorem ⟨Multivariable Circle Criterion (Hurwitz case)⟩

$A: \mathbb{R}^n \to \mathbb{R}^n$, $B: \mathbb{R}^m \to \mathbb{R}^n$, $C: \mathbb{R}^n \to \mathbb{R}^m$

Let $[A, B, C]$ be a controllable and observable triple. Let A be Hurwitz. Suppose ψ satisfies the sector condition

$$\psi^T(t,y)\,(\psi(t,y) - Ky) \leq 0 \qquad \forall\, t \geq 0,\ y \in \mathbb{R}^m$$

$K = K^T > 0$

Then the closed loop system

$$(*) \quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \\ u = -\psi(t,y) \end{cases}$$

is absolutely stable <u>provided</u>

$$Z(s) = \mathbb{1}_m + K\,G(s)$$

is strict positive real.

If the sector condition is only valid for $\Gamma \subset \mathbb{R}^p$, $0 \in \mathring{\Gamma}$, then the strict positive reality of $Z(s)$ ensures only that the closed loop system is absolutely stable with finite domain.

$\mathring{\Gamma}$ = interior of Γ

<u>Proof</u> $Z(s) = \mathbb{1}_m + K\,G(s)$ is the transfer function of the linear system

$$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$$

$$\tilde{y} = KC\tilde{x} + \tilde{u}.$$

Setting $D = D^T = \mathbb{1}_m$ in the Kalman Yacubovitch Popov lemma & also further replacing C in the lemma by KC one concludes that there exists $P = P^T > 0$ and

and $\varepsilon > 0$

matrices L and W such that

$$A^T P + PA = -L^T L - \varepsilon P$$
$$P B = (KC)^T - L^T W$$
$$W^T W = D + D^T = 2\mathbb{1}_m$$

Take $W = \sqrt{2}\, \mathbb{1}_m \Rightarrow PB = C^T K - \sqrt{2}\, L^T$.

Now consider the function

$$V(x) = x^T P x.$$

Along trajectories of the closed loop system $(*)$,

$$\begin{aligned}
\dot V &= \dot x^T P x + x^T P \dot x \\
&= (Ax - B\psi(t, Cx))^T P x + x^T P(Ax - B\psi(t,Cx)) \\
&= x^T(A^T P + PA)x - 2x^T PB\psi(t, Cx).
\end{aligned}$$

Since $-2\psi^T(t,y)(\psi(t,y) - Ky) \geq 0$ by the sector condition, it follows that

$$\begin{aligned}
\dot V &\leq x^T(A^TP + PA)x - 2x^T PB\psi - 2\psi^T(\psi - K\cancel{C}x) \\
&= x^T(A^TP + PA) + 2x^T(C^TK - PB)\psi - 2\psi^T\psi \\
&= -\varepsilon x^T P x - x^T L^T L x + 2\sqrt{2}\, x^T L^T \psi - 2\psi^T\psi \quad \text{(by KYP)} \\
&= -\varepsilon x^T P x - (Lx - \sqrt{2}\psi)^T (Lx - \sqrt{2}\psi) \\
&\leq -\varepsilon x^T P x
\end{aligned}$$

The function V satisfies all the hypotheses of the Time Varying Lyapunov Theorem (see Lecture Notes Lecture 6, part (i), page 7) with $\alpha_1(r) = \lambda_{min}(P)\, r^2$, $\alpha_2(r) = \lambda_{max}(P)\, r^2$ and $\alpha_3(r) = \varepsilon \lambda_{min}(P)\, r^2$.

$c_i = c_i r^2$
$i = 1,2,3$

So the closed loop system has the origin as an uniformly asymptotically stable (in fact exponentially stable) equilibrium point. ▨

Remark: We have used the easily shown fact that $[A, C]$ observable $\Leftrightarrow$ $[A, KC]$ is observable for K any nonsingular matrix

Remark Suppose A is not Hurwitz. It is possible that there exists a K_{min} such that the matrix $(A - BK_{min}C)$ is Hurwitz (under the assumption that $[A, B]$ is controllable and $[A, C]$ is observable). (In fact, conditions for this are difficult to determine and there is a deep problem hidden here — work of Byrnes, Brockett, Rosenthal and others. We will sweep these difficulties under the rug!)

This work involves the methods of algebraic geometry including the Schubert calculus

Now the closed loop system of

$$\dot{x} = Ax + Bu \; ; \quad y = Cx \; ; \quad u = -\psi(t, y) \quad \text{is}$$

$$\begin{aligned} \dot{x} &= Ax - B\psi(t, Cx) \\ &= (A - BK_{min}C)x - B(\psi(t, Cx) - K_{min}Cx) \end{aligned}$$

which is the closed loop system of

$$\dot{x} = (A - BK_{min}C)x + Bu \; ; \quad y = Cx \; ; \quad u = -\tilde{\psi}(t, y)$$

where,

$$\tilde{\psi}(t, y) \triangleq \psi(t, y) - K_{min}y \, .$$

Note that the triple $[A, B, C]$ is controllable and observable iff the triple $[A - BK_{min}C, B, C]$ is controllable and observable. (exercise in linear algebra)

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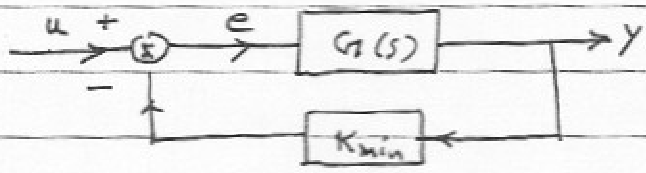
The transfer function $\tilde{G}(s)$ of the system

$$\dot{x} = (A - BK_{min}C)x + Bu$$

$$y = Cx$$

is the same as the transfer function of the closed loop system in the ~~adjoining~~ block diagram



where $G(s) = C(s\mathbb{1} - A)^{-1}B$ as before.

Observe that, in terms of Laplace transforms of inputs and outputs,

$$Y(s) = G(s)\,E(s)$$
$$E(s) = U(s) - K_{min}Y(s)$$

$$\Rightarrow \quad Y(s) = G(s)\,U(s) - G(s)K_{min}Y(s)$$

$$\Rightarrow \quad (\mathbb{1} + G(s)K_{min})Y(s) = G(s)U(s)$$

$$\Rightarrow \quad Y(s) = (\mathbb{1} + G(s)K_{min})^{-1}G(s)U(s)$$

$$\Rightarrow \quad \tilde{G}(s) = (\mathbb{1} + G(s)K_{min})^{-1}G(s).$$

On the other hand

$$E(s) = U(s) - K_{min}Y(s)$$
$$= U(s) - K_{min}G(s)E(s)$$

$$\Rightarrow \quad (\mathbb{1} + K_{min} G(s)) E(s) = U(s)$$

$$\Rightarrow \quad E(s) = (\mathbb{1} + K_{min} G(s))^{-1} U(s)$$

$$\Rightarrow \quad Y(s) = G(s) E(s)$$
$$= G(s) (\mathbb{1} + K_{min} G(s))^{-1} U(s)$$

So we have two formulas

$$\tilde{G}(s) = (\mathbb{1} + G(s) K_{min})^{-1} G(s)$$

$$= G(s) (\mathbb{1} + K_{min} G(s))^{-1}$$

(and of course ~~they~~ are equivalent as a little algebra shows).

Now $A - BK_{min}C$ is Hurwitz iff all the poles of $\tilde{G}(s) = G(s)(\mathbb{1} + K_{min} G(s))^{-1}$ are in $\mathbb{C}^-$.

Applying the sector condition to $\tilde{\psi}$,

$$\tilde{\psi}(t,y)^T (\tilde{\psi}(t,y) - K y) \leq 0$$

$$\Leftrightarrow \quad (\psi(t,y) - K_{min} y)^T (\psi(t,y) - (K_{min} + K) y) \leq 0$$

$$\Leftrightarrow \quad (\psi(t,y) - K_{min} y)^T (\psi(t,y) - K_{max} y) \leq 0$$

for $K_{max} = K_{min} + K$.

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The relevant positive real transfer function is

$$\tilde{Z}(s) = \mathbb{1} + K\,\tilde{G}(s)$$

$$= \mathbb{1} + KG\,(\mathbb{1} + K_{min}G)^{-1}$$

$$= (\mathbb{1} + K_{min}G)(\mathbb{1} + K_{min}G)^{-1} + KG(\mathbb{1} + K_{min}G)^{-1}$$

$$= (\mathbb{1} + (K_{min} + K)G)(\mathbb{1} + K_{min}G)^{-1}$$

$$= (\mathbb{1} + K_{max}G)(\mathbb{1} + K_{min}G)^{-1}$$

Now we are ready to state the multivariable circle criterion <u>without</u> the Hurwitz Assumption on A

<u>Theorem</u> (CIRCLE CRITERION) Let $[A, B, C]$ be a controllable and observable triple. Suppose ψ satisfies the sector condition

$$(\psi^{T}(t,y) - K_{min}\,y)^{T}(\psi(t,y) - K_{max}\,y) \leq 0$$

$\forall\ t \geq 0$, $\forall\ y \in \mathbb{R}^m$, and $K = K_{max} - K_{min} = K^T > 0$ given. Then the closed loop system is absolutely stable <u>provided</u>

(a) $\tilde{G}(s) = G(s)\,(\mathbb{1} + K_{min}G(s))^{-1}$ is "Hurwitz" (analytic in $\{s : Re(s) \geq 0\}$)

(b) $\tilde{Z}(s) = (\mathbb{1} + K_{max}G)(\mathbb{1} + K_{min}G)^{-1}$ is strict positive real.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$u = -\psi(t,y)$$

Proof : From the remarks preceding the above statement, it is clear that all one has to do is to appeal to the equivalences of closed loop systems with and without the loop transformation arising from the feedback $A \mapsto A - BK_{min}C$ and appeal to the Hurwitz case already proved. $\blacksquare$

Where does this name "circle criterion" come from? This is an interesting story going back to the work of ~~Hy~~ Harry Nyquist the AT&T Mathematician who investigated graphical methods for feedback amplifier stability in long-distance (transatlantic) telephony. This is a direct application of the principle of the argument in complex variable theory.

This work of Nyquist was in the 1930's. He is the same man who developed the theoretical understanding of thermal noise (Nyquist-Johnson noise) in electrical devices in parallel with the work of the experimentalist Johnson (in 1928). Extremely important to day in the context of MEMS devices

First specialize to the single input single output case.

$$\text{Let } \Pi_G \triangleq \{ u + jv = G(j\omega) \mid \omega \in \mathbb{R} \}$$
$$= \text{image of the imaginary axis under } G(\cdot)$$

be the Nyquist locus of G.

Theorem (Nyquist) Let Π_G be bounded (i.e. G is proper and no poles on the $j\omega$ axis). We will say that Π_G encircles

$G(s) = \frac{q(s)}{p(s)}$ is proper if $\deg q(s) \leq \deg p(s)$ ∴ strictly proper if ∴ $<$

a point $u_0 + jv_0$, ρ times, if $u_0 + jv_0$ is not on Γ_G and $2\pi\rho$ = net increase in the argument of $G(j\omega) - (u_0 + jv_0)$ as ω increases from $-\infty$ to $+\infty$.

clockwise encirclement $\Leftrightarrow$ direction of increasing argument

counterclockwise encirclement $\Leftrightarrow$ direction of decreasing argument.

Suppose Γ_G is bounded. If G has ν poles in the half plane $\mathbb{C}^+$ then $\frac{G}{1+kG}$ has $\rho + \nu$ poles in the half plane $\mathbb{C}^+$ if the point $-\frac{1}{k} + j0$ is not on Γ_G and Γ_G encircles it ρ times in the clockwise sense.

(proof: see Franklin et. al.) → reference list for this class.

Corollary If Γ_G is bounded and $-\frac{1}{k} + j0$ is not on Γ_G and G has ν poles in $\mathbb{C}^+$ then the feedback $u = -ky$ stabilizes the closed loop system if Γ_G encircles $-\frac{1}{k} + j0$ ν times in the counterclockwise direction

Lemma Let $g(s)$ be a scalar transfer function. Let $g(s)$ be proper (i.e. $g(s) = \frac{q(s)}{p(s)} + d$ where $\deg(q) < \deg(p)$, p monic and d a constant). poles of $g(s)$ all lie in $\mathbb{C}^-$. Then $g(s)$ is strict positive real iff

$$Re\left(g(j\omega)\right) > 0 \quad \forall\ \omega \in \mathbb{R}.$$

Proof : < See H. Khalil → page 404.

Theorem: Let $g(s)$ be a scalar transfer function $= c(s\mathbb{1}-A)^{-1}b$ $[A, b, c]$ controllable and observable. Let $\psi(t, y)$ satisfy the sector condition:

$$\alpha y^2 \leq y\psi(t, y) \leq \beta y^2$$

Then absolute stability of the closed loop system

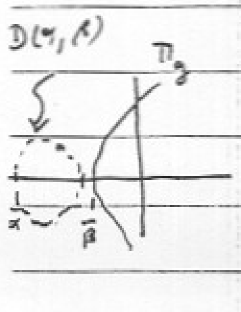
$$\dot{x} = Ax + bu$$

$$y = cx$$

$$u = -\psi(t, y)$$

holds provided one of the following condition apply

(i) If $0 < \alpha < \beta$, the Nyquist locus does not enter the disk $D(\alpha, \beta)$ and encircles it ν times in the counterclockwise direction where ν = # poles of $g(s)$ in $\mathbb{C}^+$



(ii) If $0 = \alpha < \beta$, $g(s)$ is "Hurwitz", and the Nyquist plot Γ_g lies to the right of the line $Re(s) = -\frac{1}{\beta}$

(iii) If $\alpha < 0 < \beta$, $g(s)$ is "Hurwitz" and the Nyquist plot Γ_g lies in the interior of the disk $D(\alpha, \beta)$

<u>Proof</u> : specializing the multivariable circle criterion to this case, we seek conditions to ensure that

(a) $\dfrac{g(s)}{1+\alpha g(s)}$ is Hurwitz <u>and</u>

(b) $\dfrac{1+\beta g(s)}{1+\alpha g(s)}$ is strict positive real.

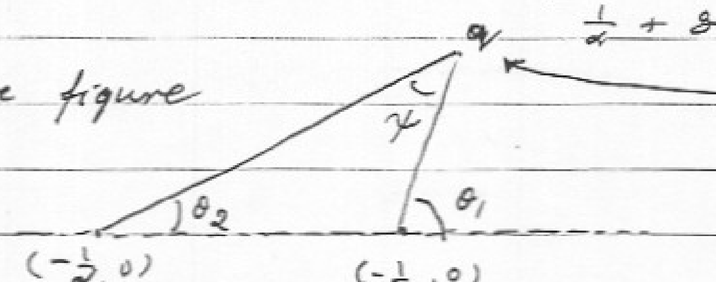
For (b) it is equivalent to check

$$Re\left(\frac{1+\beta g(j\omega)}{1+\alpha g(j\omega)}\right) > 0 \quad \forall \omega \in \mathbb{R}$$

<u>In case (i)</u> $0 < \alpha < \beta$, this is equivalent to checking

$$Re\left(\frac{\frac{1}{\beta} + g(j\omega)}{\frac{1}{\alpha} + g(j\omega)}\right) > 0 \quad \forall \omega \in \mathbb{R}$$

Consider the figure



$q \in \Gamma_g$

$$\psi = \theta_1 - \theta_2$$

$$\theta_1 = \arg\left(\frac{1}{\beta} + g(j\omega)\right)$$

$$\theta_2 = \arg\left(\frac{1}{\alpha} + g(j\omega)\right)$$

$$Re\left(\frac{\frac{1}{\beta}+g(j\omega)}{\frac{1}{\alpha}+g(j\omega)}\right) = r.\cos\psi$$

where $r \geq 0$. $\cos\psi > 0 \Leftrightarrow \psi = \theta_1 - \theta_2 < \frac{\pi}{2}$

By elementary geometry g has to lie outside $D(\alpha,\beta)$ the disk with diameter joining $(-\frac{1}{\alpha}, 0)$ and $(-\frac{1}{\beta}, 0)$. For the encirclement condition use the corollary to Nyquist.

In case (ii) the condition for strict positive reality is $Re\left(\frac{1}{\beta} + g(j\omega)\right) > 0$

$\leftrightarrow \cos(\theta_1) > 0$

$\leftrightarrow \theta_1 < \pi/2$

$\leftrightarrow \Gamma_g$ lies to the right of the vertical line $\{s: Re(s) = -\frac{1}{\beta}\}$.

In case (iii) same arguments as in (i) above but we seek $\psi > \frac{\pi}{2} \leftrightarrow \Gamma_g$ lies in the interior of the disk $D(\alpha,\beta)$.

(because α and β have opposite sign the strict positive reality condition is
$$Re\left(\frac{\frac{1}{\beta}+g(j\omega)}{\frac{1}{\alpha}+g(j\omega)}\right) < 0$$
)

∎

The Popov criterion for absolute stability is based on

(i) restrictions on ψ

(ii) use of non-quadratic Lyapunov function

(i) $\psi = \psi(y) = (\psi_1(y_1), \psi_2(y_2), \ldots, \psi_m(y_m))^T$

$\psi^T(y)(\psi(y) - Ky) \le 0 \qquad K = \mathrm{diag}(\beta_1, \ldots, \beta_m), \; \beta_i \geq 0 \; \forall i$

(ii) $$V(x) = x^T P x + 2\eta \int_0^y \sum_{i=1}^m \psi_i(\sigma_i)\, \beta_i \, d\sigma_i$$

$$\triangleq x^T P x + 2\eta \int_0^y \psi^T(\sigma) K \, d\sigma \qquad \text{here } y = Cx$$

By sector condition, $\psi(\sigma) \geq 0$ for $\sigma \geq 0$ (componentwise) $\Rightarrow \int_0^y \sum_i^m \psi_i(\sigma_i)\beta_i \, d\sigma_i \geq 0$

$\Rightarrow V(x) \geq 0$ (as long as $P = P^T > 0$).

Along trajectories of the (usual) closed loop system

$$\dot V = \dot x^T P x + x^T P \dot x + 2\eta \psi^T K \dot y \qquad (y = Cx)$$

$$= (Ax - B\psi)^T P x + x^T P (A - B\psi) + 2\eta \psi^T K C (Ax - B\psi)$$

$$= x^T (A^T P + PA) x - 2 x^T P B \psi + 2\eta \psi^T K C (Ax - B\psi)$$

Since $-2\psi^T(\psi - Ky) \geq 0$ we get,

by the sector condition

$$\dot V \le x^T(A^T P + PA)x - 2x^T PB\psi + 2\eta\psi^T KC(Ax - B\psi) - 2\psi^T(\psi - Ky)$$

$$= x^T(A^T P + PA)x - 2x^T(PB - \eta A^T C^T K - C^T K)\psi - \psi^T(2\mathbb{1} + \eta KCB + \eta B^T C^T K)\psi$$

Choose η small enough s.t.

$$2\mathbb{1} + \eta KCB + \eta B^T C^T K > 0$$

$\longleftrightarrow$ we can find W s.t.

$$W^T W = 2\mathbb{1} + \eta KCB + \eta B^T C^T K$$
$$= (\mathbb{1} + \eta KCB) + (\quad)^T$$

Suppose $P = P^T > 0$ and $\exists\ L$ and $\epsilon > 0$ s.t.

$$A^T P + PA = -L^T L - \varepsilon P$$
$$PB = C^T K + \eta A^T C^T K - L^T W$$
$$= (C + \eta CA)^T K - L^T W$$

Then

$$\dot{V}(x) \leq -\varepsilon x^T P x - x^T L^T L x + 2x^T L^T W \psi - \psi^T W^T W \psi$$
$$= -\varepsilon x^T P x - (Lx - W\psi)^T (Lx - W\psi)$$

$$\leq -\varepsilon x^T P x$$
$$< 0 \qquad x \neq 0.$$

Thus we get absolute stability. The question of P, L, ε, W is settled by the KYP lemma.

$$Z(s) = (\mathbb{1} + \eta KCB) + (KC + \eta KCA)(s\mathbb{1} - A)^{-1} B$$
$$= \mathbb{1} + \eta KC(s\mathbb{1} - A + A)(s\mathbb{1} - A)^{-1} B + KC(s\mathbb{1} - A)^{-1} B$$
$$= \mathbb{1} + \eta s KC(s\mathbb{1} - A)^{-1} B + KC(s\mathbb{1} - A)^{-1} B$$
$$= \mathbb{1} + (\eta s + 1) K G(s)$$

Suppose η is chosen that $-\frac{1}{\eta}$ is not an eigenvalue of A. Then $(A, K(C + \eta CA))$ is observable iff (A, C) is observable.

Then by KYP, P, L, ε, W exist if

$$Z(s) = \mathbb{1} + (\eta s + 1) K G(s)$$ is strict positive real. $\underline{\text{We have proved}}$

<u>THEOREM</u> (multivariable Popov criterion)

$$(*) \quad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -\psi(y) \end{aligned}$$

A Hurwitz, (A, B, C) $\left\{ \begin{array}{l} \text{controllable \&} \\ \text{observable} \end{array} \right\}$

$\psi(y) = (\psi_1(y_1), \dots, \psi_m(y_m))^T$

$K = \operatorname{diag}(\beta_1, \dots, \beta_m)$

$\beta_i > 0 \quad \forall i.$

$$0 \le y_i \psi_i(y_i) \le \beta_i y_i^2 \longleftarrow \text{(sector condition)}$$

Then $(*)$ is absolutely stable if $\exists\, \eta \geq 0$ s.t. $-\frac{1}{\eta} \in \text{spectrum}(A)$ and

$$Z(s) = \mathbb{1}_m + (1 + \eta s) K G(s)$$

is strict positive real. $\blacksquare$

Choose η small enough s.t.

$$2\mathbb{1} + \eta KCB + \eta B^T C^T K \geq 0$$

$\leftrightarrow$ we can find W s.t.

$$W^T W = 2\mathbb{1} + \eta KCB + \eta B^T C^T K$$
$$= (\mathbb{1} + \eta KCB) + (\quad\quad)^T$$

Suppose $P = P^T > 0$ and $\exists\, L$ and $\epsilon > 0$ s.t.

$$A^T P + PA = -L^T L - \varepsilon P$$
$$PB = C^T K + \eta A^T C^T K - L^T W$$
$$= (C + \eta CA)^T K - L^T W$$

Then

$$\dot{V}(x) \leq -\varepsilon x^T P x - x^T L^T L x + 2x^T L^T W \psi - \psi^T W^T W \psi$$
$$= -\varepsilon x^T P x - (Lx - W\psi)^T (Lx - W\psi)$$

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is strict positive real. $\square$

Remark (a) With $\eta = 0$, this reduces to a special case of the circle criterion

(b) With $\eta > 0$, we get absolute stability under weaker conditions (but for a restricted class of nonlinear maps ψ)

(c) For $m = 1$ (SISO case), we have a graphical test

choose η s.t. $Z(\infty) = \lim_{s \to \infty} Z(s) = W^2 > 0$.

Then $Z(s)$ is strict positive real iff

$$\operatorname{Re}\left[1 + (1 + \eta j\omega)\, k\, g(j\omega)\right] > 0 \quad \forall \omega \in \mathbb{R}$$

note $k > 0$

$$\Longleftrightarrow \quad \frac{1}{k} + \operatorname{Re}(g(j\omega)) - \eta\, \omega \operatorname{Im}(g(j\omega)) > 0 \quad \forall \omega \in \mathbb{R}$$

$\Longleftrightarrow$ Popov locus / plot lies to the right of the line that intercepts the point $-\frac{1}{k} + j0$ with slope η

Here: Popov locus $= \mathcal{P}_g = \{ u + jv : u = \operatorname{Re}(g(j\omega)),\ v = \omega \operatorname{Im}(g(j\omega)) \ \forall \omega \in \mathbb{R} \}$