Input-Output Analysis of Nonlinear Systems

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The understanding of systems from a stimulus-response or input-output (or external) point of view has a long history, pre-dating the ~~[illegible]~~ infusion of the state-space or internal descriptions. It is the natural thing to consider in exploring a wide variety of complex systems (from economics, biology as well, besides the world of technology). In some settings, definitions and theorems in the state-space point-of-view lead to corresponding results in the external point-of-view. The converse is not the case, without additional hypotheses.

To illustrate:

consider a linear time varying system

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$

$$y(t) = C(t)\,x(t)$$

Assume that

(i) the transition matrix Φ defined

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$$\dot{\Phi}(t,t_0) = A(t)\,\Phi(t,t_0)$$
$$\Phi(t_0,t_0) = \mathbb{1}$$

satisfies $\quad \|\Phi(t,t_0)\| \le m\, e^{-k(t-t_0)}$

$\forall\, t \geq t_0$, and some $k>0$, $m>0$.

(thus the system $\dot{x}(t) = A(t)x(t)$)
(has ~~uniform~~ exponential stability of the zero solution)
(— we call this <u>internal stability</u>)

(ii) $\quad \|C(t)\| \le c \qquad \|B(t)\| \le b$

$\forall\ t \geq t_0$.

The variation of constants formula tells us that

$$y(t) = C(t)\,\Phi(t,t_0)\,x_0 + \int_{t_0}^{t} C(t)\,\Phi(t,\sigma)\,B(\sigma)\,u(\sigma)\,d\sigma$$

$$\Rightarrow \quad \|y(t)\| \le c\,m\,e^{-k(t-t_0)}\,\|x_0\| + \frac{cbm}{k}\left(1-e^{-k(t-t_0)}\right)\|u\|$$

where we assume bounded inputs:

$$\|u(t)\| \le \sup_{t\geq t_0}\|u(t)\| \triangleq \|u\| < \infty$$

$$\Rightarrow \quad \|y(t)\| \le \beta + \gamma\,\|u\|$$

where $\beta = cm\|x_0\|$ & $\gamma = \dfrac{cbm}{k}$

Internal stability + Bounded Inputs $\Rightarrow$ Bounded Outputs

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The property of bounded inputs always giving rise to bounded outputs is a type of external stability. As can be seen from the example below

$$\dot{x}_1 = -x_1 + u$$

$$\dot{x}_2 = x_1^2 + x_2^2$$

$$y = x_1$$

external stability $\not\Longrightarrow$ internal stability (of the zero solution)

We would like to state and prove certain basic notions & theorems of <u>external stability</u>, connect them to interesting physical properties of systems and establish ties to notions of internal ~~ex~~ stability. ~~A~~ The initial steps in this direction include:

(a) proper definitions of function spaces of input and output signals

(b) concepts of causality, feedback, well-posedness and passivity (c): various stability and finite gain theorems.

The signals applicable in the present context cannot be of the finite energy over the infinite time interval $[0, \infty)$. Think of ramp signals.

<u>Definition</u> The truncation operator $(\cdot)_T$ on functions on $[0, \infty)$ is defined by

$$x_T(t) = \begin{cases} x(t) & t \leq T \\ 0 & t > T \end{cases}$$

for $T \geq 0$.

<u>Definition</u> The space L_{pe} is defined by

$$L_{pe} = \{ x(\cdot) : [0, \infty) \to \mathbb{R} \,/\, x_T \in L_p \,,\ \forall\, T \geq 0 \}$$

<u>example</u>: $x(t) = t \quad t \geq 0$

$x(\cdot) \notin L_p$ for any $p \in [1, \infty)$

But $x_T(\cdot) \in L_p \quad \forall\, T \geq 0$.

<u>Lemma</u>: For each $p \in [1, \infty]$, the set $L_{pe}[0, \infty]$ is a linear space. If $p \in [1, \infty]$ and $f \in L_{pe}[0, \infty)$, then

(i) $\| f_T(\cdot) \|$ is a nondecreasing function of T

(ii) $f \in L_p[0, \infty)$ iff there exists a finite constant m such that $\| f_T \| \leq m \quad \forall\, T > 0$. In that case, $\| f \|_p = \lim_{T \to \infty} \| f_T \|_p$ $\boxed{\text{PROOF} \to \text{EXERCISE}}$

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<u>Remark</u>: $L_{pe}[0,\infty)$ itself does <u>not</u> carry a norm, that agrees with the norm in $L_p[0,\infty)$ when restricted to that subspace.

$$L_p^r = \underbrace{L_p \times L_p \times \cdots \times L_p}_{r \text{ times}}$$

i.e each function $f \in L_p^r$ is an $\mathbb{R}^r$ valued function characterized by, each component $f_i \in L_p$. Similarly for L_{pe}^r.

<u>Definition</u> <<u>Causality</u>>

$F : L_{pe}^m \to L_{pe}^q$ is said to be a <u>causal</u> map/system if

$$(F(u))_T = (F(u_T))_T \qquad \forall\ T \geq 0 \text{ and } \forall\ u \in L_{pe}^m$$

<u>Lemma</u> A map/system $F : L_{pe}^m \to L_{pe}^q$ is causal iff whenever $u_1, u_2 \in L_{pe}^m$ and $(u_1)_T = (u_2)_T$ for some $T < \infty$, we have $(F(u_1))_T = (F(u_2))_T$

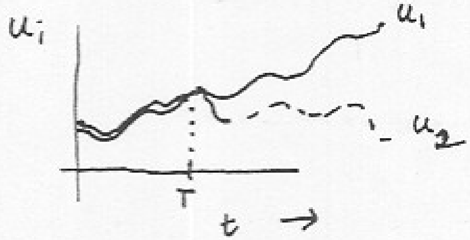
<u>Proof</u> ($\Rightarrow$) suppose F satisfies the condition in the statement. Let $u \in L_{pe}^m$. Let $T < \infty$ be arbitrary. Then $(u)_T = (u_T)_T$. By hypothesis,

$$(F(u))_T = (F(u_T))_T.$$

Since T is arbitrary, we have established causality.

($\Leftarrow$) assume F is causal.

Let $u_1, u_2 \in L_{pe}^m$ be such that for some $T > 0$, $(u_1)_T = (u_2)_T$



Now $(F(u_1))_T = (F(u_{1T}))_T$
$= (F(u_{2T}))_T$
$= (F(u_2))_T$ ☒

Stability. in the external sense

Definition (1) A map/system $F : L_{pe}^m \to L_{pe}^q$ is said to be stable if there exist finite constants $\gamma, \beta > 0$ such that

$$\|(F(u))_T\| \le \gamma \|u_T\| + \beta$$

$\forall\ u \in L_{pe}^m$ and $\forall\ T \geq 0$.

gain = smallest such γ
offset = " " β

Definition (1′) A map/system $F: L_{pe}^m \to L_{pe}^q$ is said to be stable if

(i) $F(u) \in L_p^q$ whenever $u \in L_p^m$ and in that case

(ii) there exist constants $\gamma, \beta > 0$ s.t.

$$\|F(u)\| \le \gamma \|u\| + \beta \qquad \forall\ u \in L_p$$

Remark The two definitions (1) & (1′) are equivalent.

Small Gain Theorem

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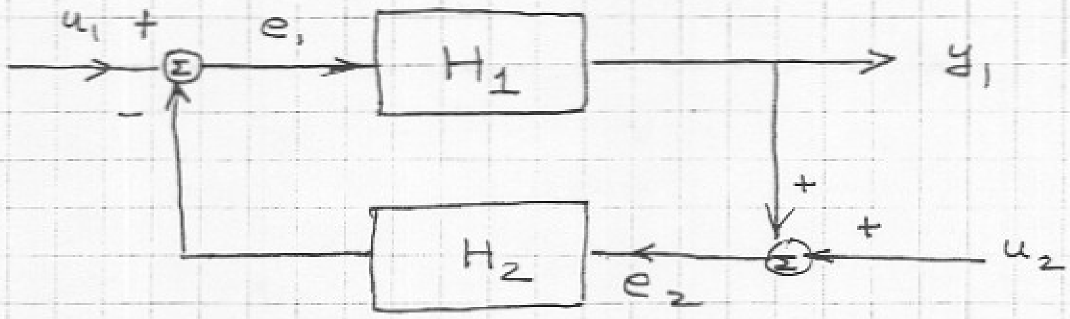

Assume

(H1) The maps $H_1 : L^m_{pe} \to L^q_{pe}$

$H_2 : H^q_{pe} \to L^m_{pe}$

are <u>causal</u>.

(H2) ~~suppose~~ H_i are stable with <u>gains</u> γ_i

and <u>offsets</u> β_i satisfying

$$\| H_i(u)_T \| \leq \gamma_i \| u_T \| + \beta_i$$

$i = 1, 2$.

(H3) For every pair of inputs $u_1 \in L^m_{pe}$ and $u_2 \in L^q_{pe}$, there exist unique outputs $e_1 \in L^m_{pe}$ and $e_2 \in L^q_{pe}$ [Well-posedness]

If further $\gamma_1 \gamma_2 < 1$, then

(a) $\forall\, u_1 \in L^m_{pe} \quad u_2 \in L^q_{pe}$

$$\| e_{1T} \| \leq \frac{1}{1-\gamma_1\gamma_2} \left(\| u_{1T} \| + \gamma_2 \| u_{2T} \| + \beta_2 + \gamma_2 \beta_1 \right)$$

$$\| e_{2T} \| \leq \frac{1}{1-\gamma_1\gamma_2} \left(\| u_{2T} \| + \gamma_1 \| u_{1T} \| + \beta_1 + \gamma_1 \beta_2 \right)$$

$\forall\, T \geq 0$

and (b) if $u_1 \in L_p^m$ and $u_2 \in L_p^q$, then $e_1, y_2 \in L_p^m$ and $e_2, y_1 \in L_p^q$, and the norms of e_1 and e_2 are bounded above by the r.h.s. in (a) with nontruncated functions.

<u>Proof</u>

Below, we will use causality freely to write $\| F(u_T)_T \| = \| F(u)_T \|$ as needed.

By hypothesis (H3) we can solve uniquely for $e_{1T} = u_{1T} - (H_2(e_{2T}))_T$ and

$$e_{2T} = u_{2T} + (H_1(e_{1T}))_T .$$

Then
$$\begin{aligned}
\|e_{1T}\| &\leq \|u_{1T}\| + \|H_2(e_{2T})_T\| \\
&\leq \|u_{1T}\| + \gamma_2 \|e_{2T}\| + \beta_2 \\
&= \|u_{1T}\| + \gamma_2 \|u_{2T} + H_1(e_{1T})_T\| + \beta_2 \\
&\leq \|u_{1T}\| + \gamma_2 \|u_{2T}\| + \gamma_2\gamma_1 \|e_{1T}\| \\
&\quad + \gamma_2 \beta_1 + \beta_2
\end{aligned}$$

Since $\gamma_1 \gamma_2 < 1$ we can write

$$\| e_{1T} \| \leq \frac{1}{1-\gamma_1\gamma_2} \left(\|u_{1T}\| + \gamma_2 \|u_{2T}\| + \gamma_2\beta_1 + \beta_2 \right)$$

Similarly for $\|e_{2T}\|$.

Rest is straightforward!

If $u_1 \in L_p^m$ and $u_2 \in L_p^q$ then,

$$\|u_{1T}\| \leq \|u_1\| \quad \forall\ T \geq 0 \quad \text{and}$$

$$\|u_{2T}\| \leq \|u_2\| \quad \forall\ T \geq 0.$$

Hence $\|e_{iT}\|$ is bounded uniformly in $T \Rightarrow e_1 \in L_p^m$ and $e_2 \in L_p^q$.

$$\|y_{1T}\| \leq \gamma_1 \|e_{1T}\| + \beta_1 \qquad \forall\ T \geq 0$$

$$\leq \gamma_1 \|e_1\| + \beta \qquad \text{uniformly in } T.$$

$\Rightarrow y_1 \in L_p^q$. Similarly $y_2 \in L_p^m$ ▨

<u>Remark</u> We interpret the above result as saying that the feedback system is stable if $\gamma_1 \gamma_2 < 1$.

In the small gain theorem the well-posedness hypothesis H3 appears to be hard to verify. One would like a sufficient condition that would be strong enough to imply this.

The assumption of a stronger hypothesis can ensure that hypothesis H3 on well-posedness holds in fact.

Definition

A map $F : L_{pe}^{m} \to L_{pe}^{q}$ is said to be incrementally finite gain stable if

(i) . $F(0) \in L_p^q$ where 0 is the identically zero input.

(ii) For all $T > 0$, $u, v \in L_{pe}^{m}$, there exists $k > 0$ such that

$$\| F_T(u) - F_T(v) \| \leq k \, \| u_T - v_T \|$$

(k is independent of T, u, v etc.)

Lemma: If $F : L_{pe}^{m} \to L_{pe}^{m}$ is causal and incrementally finite gain stable with gain $k < 1$, then there is a unique $u^* \in L_{pe}^{m}$ such that $F(u^*) = u^*$.

Proof By hypothesis,

$$\| F_T(u) - F_T(v) \| \leq k \, \| u_T - v_T \|$$

$\forall \; u, v \in L_{pe}^{m}$, $\forall \; T > 0$ and $k < 1$.

By causality $F_T(u) = F_T(u_T)$.

Hence, $\|F_T(u_T) - F_T(v_T)\| \le k \|u_T - v_T\| \quad \forall\, T \geq 0$

$$\text{But } \|F(u) - F(v)\| \le \sup_{T\geq 0} \|F_T(u) - F_T(v)\|$$

$$< k \sup_{T\geq 0} \|u_T - v_T\|$$

$$= k \|u - v\| \qquad \forall\, u, v \in L_p^m$$

Thus $F : L_p \to L_p$ the restriction to L_p, is a global contraction. Since L_p is a Banach space, there is a unique fixed point $u^* \in L_p$ such that

$$F(u^*) = u^*$$

(we can compute u^* by successive approximation algorithm initialized in L_p).

Can there be a $v^* \in L_{pe}$, but $v^* \notin L_p$ such that $F(v^*) = v^*$ (and $v^* \neq u^*$ necessarily) ?

Cleanup the argument in Theorem 4.17 (Sastry)

Some examples

1. $H: L_{\infty e} \to L_{\infty e}$

$$u \mapsto u^2$$

is causal but unstable.

2 $$H_1(u)(t) = \int_0^t e^{-a(t-\tau)} u(\tau)\,d\tau$$

$$H_2(u)(t) = k\,u(t) \qquad a > 0$$

$H_1 : L_{\infty e} \to L_{\infty e}$

$$\gamma_1 = \frac{1}{a} \;;\quad \beta_1 = 0$$

$H_2 : L_{\infty e} \to L_{\infty e}$

$$\gamma_2 = |k| \qquad \beta_2 = 0$$

Small gain theorem says

$$\frac{1}{a}|k| < 1 \quad \Rightarrow \quad \text{stability of closed loop system.}$$

$$\Updownarrow$$

$$-a < k < a$$

This is conservative in the sense that $-a < k$ is a necessary and sufficient condition for closed loop stability.

(from the transfer function

$$g_{\text{closed loop}}(s) = \frac{1}{s+a+k}$$