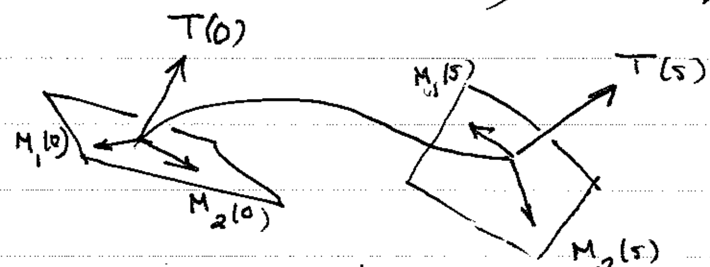
<u>Alternative way to frame a curve</u>
(ref: R. L. Bishop, American Math Monthly <u>82</u> (3): 246–251, 1975)



s = arc length parameter

At $s=0$, let $\overset{\perp}{T(0)}$ denote plane <u>normal</u> to unit tangent vector $T(0)$; similarly $\overset{\perp}{T(s)}$ is the normal plane at s. Pick a basis $\{M_1(0), M_2(0)\}$ for $\overset{\perp}{T(0)}$ such that $\{T(0), M_1(0), M_2(0)\}$ constitutes a right handed orthonormal triad. Our goal is to propagate this triad to $\{T(s), M_1(s), M_2(s)\}$ in such a way that certain <u>natural conditions are satisfied</u>:

Right handedness $\leftrightarrow$ $M_2(0) = T(0) \times M_1(0)$ and
$M_2(s) = T(s) \times M_1(s)$.

$T(s)\cdot T(s) \equiv 1 \Rightarrow T'(s) \in \overset{\perp}{T(s)}$. Hence there must exist $k_1(s)$ and $k_2(s)$ such that
$T'(s) = k_1(s)\, M_1(s) + k_2(s)\, M_2(s)$.
We call $k_1(s)$ and $k_2(s)$ (natural) curvatures,

Let $M(s)$ be any unit <u>normal field</u> along γ.

<u>Definition 1</u> We say $M(s) \in T^{\perp}(s)$ is a <u>relatively parallel field</u> along γ provided

$$M'(s) = f(s)\,T(s)$$

i.e. the vector $M(s)$ <u>turns as little as possible</u>. $\boxtimes$

(This is the natural condition we mentioned.)

We propagate $M_1(0)$, $M_2(0)$ along γ such that they remain relatively parallel at each s.

Thus,

$$M_1'(s) = f_1(s)\,T(s)$$

$$M_2'(s) = f_2(s)\,T(s).$$

for some as yet undetermined f_1, f_2.

But $M_1(s) \cdot T(s) \equiv 0$ (normality)

$$\Rightarrow \quad M_1'(s) \cdot T(s) = -M_1(s) \cdot T'(s)$$

$$= -M_1(s) \cdot \left(k_1(s) M_1(s) + k_2 M_2(s)\right)$$

$$= -k_1(s)$$

(since $M_1 \cdot M_1 \equiv 1$ and $M_1 \cdot M_2 \equiv 0$)

$$\Rightarrow \quad f_1(s) = -k_1(s)$$

Similarly $f_2(s) = -k_2(s)$.

<u>Definition 2</u> An orthonormal triad $\{T(s), M_1(s), M_2(s)\}$ is a <u>relatively parallel adapted frame</u> (RPAF) along γ provided there exist curvature functions $k_1(\cdot)$, $k_2(\cdot)$ such that

$$T' = k_1 M_1 + k_2 M_2$$

$$M_1' = -k_1 T$$

$$M_2' = -k_2 T \qquad \boxtimes$$

RPAF's are also called <u>natural Frenet frames</u>.

<u>Theorem 3</u> Given a C^2 curve γ and a choice $M_1(0)$, $M_2(0)$ in $T(0)^{\perp}$ such that $\{T(0), M_1(0), M_2(0)\}$ is a right-handed orthonormal triad, there is a unique RPAF along γ that agrees with the initial choice.

<u>Proof</u> Integrating $M_1'(s) = -k_1(s)\, T(s)$ on both sides,

$$M_1(s) = M_1(0) - \int_0^s k_1(\sigma)\, T(\sigma)\, d\sigma$$

Taking the dot product with $M_1(s)$ of both sides of

$$T'(s) = k_1(s)\, M_1(s) + k_2(s) M_2(s),$$

we get

$$k_1(s) = T'(s) \cdot M_1(s)$$

$$= T'(s) \cdot M_1(0) - \int_0^s k_1(\sigma)\, T'(s) \cdot T(\sigma)\, d\sigma$$

$$\Leftrightarrow \boxed{k_1(s) = \gamma''(s)\cdot M_1(0) - \int_0^s k_1(\sigma)\ \gamma''(s)\cdot\gamma'(\sigma)\,d\sigma}$$

Similarly $\boxed{k_2(s) = \gamma''(s)\cdot M_2(0) - \int_0^s k_2(\sigma)\ \gamma''(s)\cdot\gamma'(\sigma)\,d\sigma}$

Given the curve γ, we have two integral equations for k_i, $i=1,2$. By the standard theory of Volterra integral equations, there exist unique k_i, $i=1,2$.

Now integrate

$$\frac{d}{ds}\begin{bmatrix} T(s) & M_1(s) & M_2(s)\end{bmatrix} = \begin{bmatrix} T(s) & M_1(s) & M_2(s)\end{bmatrix}\begin{bmatrix} 0 & -k_1(s) & -k_2(s) \\ k_1(s) & 0 & 0 \\ k_2(s) & 0 & 0 \end{bmatrix}$$

starting from $\begin{bmatrix} T(0) & M_1(0) & M_2(0)\end{bmatrix} \in SO(3)$, to obtain a unique RPAF. ▨

<u>Relation to Frenet-Serret Frame</u> (4) The normal $N(s)$ and binormal $B(s)$, when defined, exist in the plane $T^{\perp}(s)$ spanned by $M_1(s)$ and $M_2(s)$.

$$N(s) = \frac{1}{\kappa(s)}\, T'(s)$$

$$= \frac{1}{\kappa(s)}\left[k_1(s)\, M_1(s) + k_2(s)\, M_2(s)\right]$$

$$\Rightarrow \quad 1 \equiv N(s)\cdot N(s) = \left(k_1^2(s) + k_2^2(s)\right) / \kappa^2(s)$$

$$\Rightarrow \quad \kappa(s) = \sqrt{k_1^2(s) + k_2^2(s)}$$

$$B(s) = T(s) \times N(s)$$

$$= T(s) \times \left(\frac{k_1(s)}{\kappa(s)} M_1(s) + \frac{k_2(s)}{\kappa(s)} M_2(s) \right)$$

$$= \frac{1}{\kappa}\left(-k_2 M_1 + k_1 M_2\right)$$

Torsion

$$\tau(s) = -B'(s)\cdot N(s)$$

$$= -\left(-\frac{k_2}{\kappa} M_1 + \frac{k_1}{\kappa} M_2\right)' \cdot \left(\frac{k_1}{\kappa} M_1 + \frac{k_2}{\kappa} M_2\right)$$

$$= \left(\frac{k_2'}{\kappa} M_1 - \frac{k_1'}{\kappa} M_2 + \frac{k_2}{\kappa} M_1' - \frac{k_1}{\kappa} M_2' + k_2 M_1 \left(\frac{1}{\kappa}\right)' - k_1 M_2 \left(\frac{1}{\kappa}\right)' \right)\cdot\left(\quad \right)$$

$$= \frac{k_2' k_1 - k_1' k_2}{\kappa^2}$$

$$= \left(\tan^{-1}\left(\frac{k_2}{k_1}\right)\right)'$$

$$= \theta'$$

where θ = polar angle in (k_1, k_2) plane, (well defined when $\kappa > 0$) also called the <u>normal development plane</u>.

Integrating, $\theta(s) = \theta(0) + \int_0^s \tau(\sigma)\,d\sigma$

Since,

$$N(s) = \cos(\theta(s))\, M_1(s) + \sin(\theta(s))\, M_2(s)$$

$$B(s) = -\sin(\theta(s))\, M_1(s) + \cos(\theta(s))\, M_2(s),$$

it is clear that $\theta(s)$ is ~~the~~ accumulated rotation (phase-shift) of $\{N(s), B(s)\}$ relative to $\{M_1(s), M_2(s)\}$. ▨

<u>Definition 5</u> One gets a picture of a curve γ in $\mathbb{R}^3$ by looking at its <u>normal development</u> $s \mapsto (k_1(s), k_2(s))$.

<u>Example 6</u> Curve γ lies in a plane $\mu^\perp$ perpendicular to a fixed vector μ. <u>We do not assume</u> $\mu^\perp$ passes through the origin.

$$\gamma(s) \cdot \mu \equiv c \quad \text{a constant.}$$

$$\Rightarrow \quad T \cdot \mu \equiv 0$$

and $\quad T' \cdot \mu \equiv 0$.

From the last equation

$$k_1(s)\,(M_1(s)\cdot\mu) + k_2(s)\,(M_2(s)\cdot\mu) \equiv 0.$$

On the other hand,

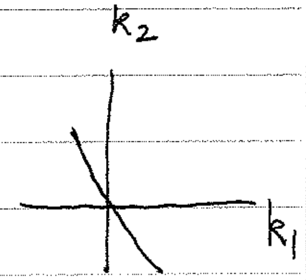
$$M_1'(s)\cdot\mu = -k_1(s)\, T(s)\cdot\mu$$
$$\equiv 0$$

$$\Rightarrow \quad M_1(s)\cdot\mu \equiv \text{constant} \triangleq a_1$$

similarly

$$M_2(s)\cdot\mu \equiv \text{constant} \triangleq a_2$$

Thus normal development $\longrightarrow$



satisfies

$$a_1\, k_1(s) + a_2\, k_2(s) = 0,$$

is on a line passing through the origin.

Normal development contains <u>no information</u> on C.

Example 7 Curve $s \mapsto \gamma(s)$ is confined to a sphere centered at p and of radius $R > 0$. Thus

$$(\gamma(s) - p)\cdot(\gamma(s) - p) \equiv R^2$$

Differentiating

$$(\gamma(s) - p)\cdot\gamma'(s) \equiv 0$$

$$\Leftrightarrow \quad (\gamma - p)\cdot T \equiv 0$$

Differentiating again

$$\gamma' \cdot T + (\gamma - p) \cdot T' \equiv 0$$

$$\Leftrightarrow \quad T \cdot T + (\gamma - p) \cdot T' \equiv 0 \qquad (\text{def}^{n} \text{ of } T)$$

$$\Leftrightarrow \quad (\gamma - p) \cdot T' \equiv -1 \qquad (T \text{ unit vector})$$

$$\Leftrightarrow \quad k_1 (\gamma - p) \cdot M_1 + k_2 (\gamma - p) \cdot M_2 \equiv -1 \qquad (T' \text{ eq}^{n})$$

Also,

$$M_i' \cdot (\gamma - p) = -k_i T \cdot (\gamma - p)$$

$$\equiv 0 \qquad i = 1, 2.$$

Now

$$(M_i \cdot (\gamma - p))' = M_i' \cdot (\gamma - p) + M_i \cdot \gamma'$$

$$= M_i' \cdot (\gamma - p) + M_i \cdot T$$

$$\equiv 0$$

Thus $\quad M_i \cdot (\gamma - p) \equiv \text{constant} = a_i \qquad i = 1, 2.$

Thus $\quad a_1 k_1(s) + a_2 k_2(s) \equiv -1$

Normal development is on a line not passing through the origin, at a distance $= \dfrac{1}{\sqrt{a_1^2 + a_2^2}}$ from the origin.

We can always write

$$(\gamma - p) = ((\gamma - p)\cdot M_1)\, M_1 + ((\gamma - p)\cdot M_2)\, M_2 + ((\gamma - p)\cdot T)\, T$$

where $(\gamma - p)\cdot M_1 \equiv a_1$; $(\gamma - p)\cdot M_2 \equiv a_2$

and $(\gamma - p)\cdot T \equiv 0$.

$$\Rightarrow R^2 = (\gamma - p)\cdot(\gamma - p)$$

$$= a_1^2 + a_2^2$$

Thus the normal development is a line <u>not</u> passing through the origin, at a distance $= \frac{1}{R}$ from the origin.

Normal development, being fully Euclidean invariant, contains no information about the center p of the sphere.

As $R \to \infty$, sphere $\to$ plane and above line $\to$ line passing through origin.

<u>Remark 8</u> Recall that the RPAF is determined upto a choice of initial orthonormal base $\{M_1(0), M_2(0)\}$. A change of basis is simply a rotation of $\{M_1(0), M_2(0)\}$

through an angle ϕ. How does this affect the curvatures k_1 and k_2?

$$\text{Let} \quad \tilde{M}_1(0) = M_1(0)\cos(\phi) - M_2(0)\sin(\phi)$$

$$\tilde{M}_2(0) = M_1(0)\sin(\phi) + M_2(0)\cos(\phi)$$

Then it can be shown by substitution in the Volterra integral equations for curvatures (see page 4) that

$$\tilde{k}_1(s) = \cos(\phi)\, k_1(s) - \sin\phi\, k_2(s)$$

$$\tilde{k}_2(s) = \sin(\phi)\, k_1(s) + \cos(\phi)\, k_2(s)$$

This corresponds to a rotation by ϕ in the normal development plane.
A curve γ determines the normal development upto such a rotation.

Example 9 Let $s \mapsto \gamma(s)$ be curve confined to a sphere $(\gamma - p)\cdot(\gamma - p) = R^2$, centered at $p \in \mathbb{R}^3$, of radius R.
For $s = 0$ pick $M_1(0) = \dfrac{\gamma(0) - p}{R}$.
Clearly $M_1(0) \in T(0)^{\perp}$ by hypothesis.

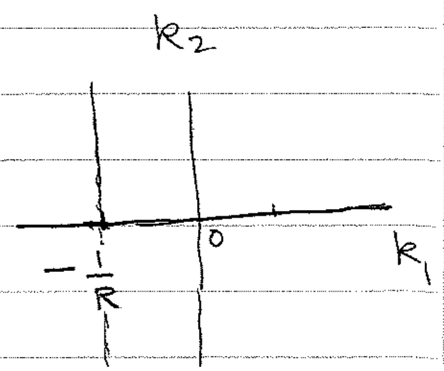
We let $M_2(0) = T(0) \times M_1(0)$ to make up the initial, right-handed orthonormal frame $\{T(0), M_1(0), M_2(0)\}$.

Now
$$a_1 = (\gamma(s) - p) \cdot M_1(s)$$
$$= (\gamma(0) - p) \cdot M_1(0)$$
$$= (\gamma(0) - p) \cdot \frac{(\gamma(0) - p)}{R}$$
$$= R$$

$$a_2 = (\gamma(s) - p) \cdot M_2(s)$$
$$= (\gamma(0) - p) \cdot M_2(0)$$
$$= (\gamma(0) - p) \cdot \left(T(0) \times \frac{(\gamma(0) - p)}{R}\right)$$
$$= 0$$

The normal development equation becomes

$$-1 \equiv a_1 \, k_1(s) + a_2 \, k_2(s)$$
$$\equiv R \, k_1(s) + 0$$
$$\Longrightarrow \quad k_1(s) \equiv -1/R$$



12

Thus the evolution equation of an RPAF for a curve confined to a sphere of radius R (centered at p) can always be taken to be of the form:

$$[T' \; M_1' \; M_2'] = [T \; M_1 \; M_2] \begin{bmatrix} 0 & 1/R & -k_2(s) \\ -1/R & 0 & 0 \\ k_2(s) & 0 & 0 \end{bmatrix}$$

with,

$$M_1(0) = \frac{\gamma(0) - p}{R} \quad ; \quad M_2(0) = T(0) \times M_1(0)$$

Since $(\gamma(s) - p) \cdot M_1(s) \equiv R$,

$$\frac{\gamma(s) - p}{R} \cdot M_1(s) \equiv 1$$

Let $M_1(s) = \dfrac{\gamma(s) - p}{R} + \delta(s)$.

Then $$\frac{\gamma(s) - p}{R} \cdot \frac{\gamma(s) - p}{R} + \frac{\gamma(s) - p}{R} \cdot \delta(s) \equiv 1$$

$$\Leftrightarrow \quad 1 + \frac{\gamma(s) - p}{R} \cdot \delta(s) \equiv 1$$

$$\Leftrightarrow \quad \frac{\gamma(s) - p}{R} \cdot \delta(s) \equiv 0$$

Then $$M_1(s) \cdot M_1(s) = \left(\frac{\gamma(s) - p}{R} + \delta(s)\right) \cdot \left(\frac{\gamma(s) - p}{R} + \delta(s)\right)$$

$$= 1 + \delta(s) \cdot \delta(s)$$

But $M_1(s)$ is a unit vector for each s.

Hence $\delta(s) \cdot \delta(s) \equiv 0 \quad \Rightarrow \quad \delta(s) \equiv 0$

$\Rightarrow \quad M_1(s) \equiv (\gamma(s) - p)/R$.

Thus, if $M_1(0) = \frac{(\gamma(0) - p)}{R}$, $\quad M_1(s) \equiv \frac{(\gamma(s) - p)}{R} \ \forall s$.

Thus M_1 is the outward normal always.